EUROPEAN CENTRAL BANK
EUROSYSTEM

Working Paper Series

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Strike while the iron is hot – optimal monetary policy under state-dependent pricing

ChaMP

No 3068

Challenges for Monetary Policy Transmission in a Changing World Network (ChaMP)

This paper contains research conducted within the network “Challenges for Monetary Policy Transmission in a Changing World Network” (ChaMP). It consists of economists from the European Central Bank (ECB) and the national central banks (NCBs) of the European System of Central Banks (ESCB).

ChaMP is coordinated by a team chaired by Philipp Hartmann (ECB), and consisting of Diana Bonfim (Banco de Portugal), Margherita Bottero (Banca d’Italia), Emmanuel Dhyne (Nationale Bank van België/Banque Nationale de Belgique) and Maria T. Valderrama (Oesterreichische Nationalbank), who are supported by Melina Papoutsi and Gonzalo Paz-Pardo (both ECB), 7 central bank advisers and 8 academic consultants.

ChaMP seeks to revisit our knowledge of monetary transmission channels in the euro area in the context of unprecedented shocks, multiple ongoing structural changes and the extension of the monetary policy toolkit over the last decade and a half as well as the recent steep inflation wave and its reversal. More information is provided on its website.

Abstract

We characterize optimal monetary policy under state-dependent pricing. The framework gives rise to nonlinear inflation dynamics: The flexibility of the price level increases after large shocks due to an endogenous rise in the frequency of price changes. In response to large cost-push shocks, optimal policy leverages the lower sacrifice ratio to curb inflation. When faced with total factor productivity shocks, an efficient disturbance, the optimal policy commits to strict price stability. The optimal long-run inflation rate is just above zero.

JEL codes: E31, E32, E52

Keywords: State-dependent pricing, large shocks, nonlinear Phillips curve, optimal monetary policy

Non-Technical Summary

Recent global economic developments, particularly the surge in inflation following the COVID-19 pandemic, have presented significant challenges for central banks. Traditional models of inflation, which assume a linear and stable relationship between inflation and economic activity, have struggled to explain these developments. This paper addresses a key question: How should central banks adjust their monetary policies when the relationship between inflation and economic output becomes more complex, particularly during periods of large economic shocks?

The paper argues that in periods of significant economic shocks – such as sudden increases in costs – firms tend to change their prices more frequently. This behavior leads to what is known as a "nonlinear Phillips curve," where inflation becomes more sensitive to changes in economic output. The central finding is that, in such scenarios, the costs of reducing inflation through tighter monetary policy are lower than previously thought because the economy adjusts more quickly when prices are already changing frequently.

The paper also demonstrates that in response to productivity shocks, the best policy is for central banks to maintain stable prices. This finding aligns with traditional models and supports the idea that price stability should remain a core focus of monetary policy during these types of shocks.

The research uses a theoretical model that incorporates state-dependent pricing—where firms' decisions to adjust prices depend on the economic environment. The model is calibrated using data from the United States. By comparing this model to a traditional linear model, the authors are able to highlight the differences in optimal policy responses under different economic conditions.

The findings have important implications for central banks. During periods of high inflation, where firms frequently adjust their prices, central banks might consider a more aggressive stance against inflation. This is because the economic cost of such policies – measured in terms of reduced output or increased unemployment – may be lower than in periods of stable prices. During productivity shocks, maintaining price stability should also remain the priority.

In conclusion, this paper provides new insights into how central banks can optimize their monetary policy in a world where the relationship between inflation and economic activity is not straightforward. By recognizing when firms are likely to adjust prices more frequently, central banks can better tailor their policies to minimize economic disruptions and maintain stable inflation.

1. Introduction

What is the optimal design of monetary policy? The traditional answer offered by the New Keynesian literature relies on the price setting model by Calvo (1983), which disregards endogenous variation in the frequency of price changes: firms update prices at random times irrespective of macroeconomic conditions.[1] In contrast, a rapidly growing literature on state-dependent price setting, sometimes referred to as "menu cost models", recognizes that firms decide when to adjust prices endogenously, taking into account monetary policy. High inflation periods, such as the recent inflation surge episode, have forcefully illustrated that the frequency of price changes is indeed responsive to macroeconomic conditions: in the U.S., it more than doubled at the 2022 inflation peak.[2] Yet, the normative aspects of these state-dependent models have received limited attention. To bridge this important gap, our paper characterizes optimal monetary policy under commitment in the canonical menu cost model of Golosov and Lucas (2007) and shows robustness in the CalvoPlus model of Nakamura and Steinsson (2010).

Our analysis arrives at a novel insight: optimal policy leans against inflation disproportionately strongly in response to large cost-push shocks, which exert upward pressure on the repricing frequency – a "strike while the iron is hot" policy. The reasons are twofold. First, the cost of the anti-inflationary policy in terms of output is smaller when the frequency of price changes increases in response to the shocks – as the price level becomes more flexible, the sacrifice ratio falls. Second, as we explain below, in our state-dependent framework the relative importance of inflation versus output in the central bank's objective stays close to that in the Calvo model. At the same time, as we show analytically, optimal policy requires full inflation stabilization after total factor productivity shocks – a version of the "divine coincidence" result after efficiency shocks, as in the canonical Calvo model.

Our baseline state-dependent price setting model closely follows the seminal paper of Golosov and Lucas (2007). In the model, a representative household consumes a continuum of differentiated goods and supplies labor in a centralized, frictionless market. Each consumption good is produced by a single firm with labor as the only input. Production technology is subject to aggregate productivity and cost-push shocks, and idiosyncratic quality shocks.[3] Firms must incur a small, fixed "menu cost" to adjust their prices. Thus, firms' pricing decisions are characterized by an (S, s) rule: When prices are within an endogenous band around the optimal reset price, firms keep them constant; otherwise, they pay the menu cost and update their price. The central bank sets the nominal interest rate.

[1] Woodford (2003); Galí (2008)

[2] See Montag and Villar (2023). The empirical relationship between inflation and frequency has been well established both in the U.S. and on other countries. The literature review below lists references.

[3] We depart from the Golosov and Lucas (2007) model in this regard, which, instead of idiosyncratic quality shocks, assumes productivity shocks. This facilitates the computation, while its implications are innocuous (see also Midrigan 2011; Alvarez et al. 2021).

We study the optimal design of monetary policy in this model. To this end, we propose a new algorithm to solve the Ramsey problem nonlinearly, so that it is suitable for assessing the impact of large aggregate shocks. In particular, we approximate the value and distribution functions over the endogenously determined relevant range and solve the set of equilibrium conditions under perfect foresight over the sequence space. We calibrate the model parameters to match the monthly frequency of price changes in the U.S. before the inflation surge[4], as well as a 20% frequency level accompanying a 10% inflation rate as experienced during the inflation surge of 2022-2023 (Montag and Villar 2023). We contrast the implications of our state-dependent model with those of a time-dependent Calvo model.[5]

The model economy is subject to three welfare-relevant distortions. The first two are caused by actual markups deviating from the efficient markup: the first distortion is caused by the *average* markup, and the second by the *dispersion* of markups. The third distortion is the resource costs related to price adjustment. The distortion caused by average markup is conventional and is present both in our baseline model and in the canonical Calvo model.[6] It incentivizes the central bank to minimize the variation, caused by aggregate shocks, in the average markup, which it can affect due to price rigidities. The behavior of the second and third distortions is distinct in the two frameworks. In our framework, aggregate shocks can reduce markup dispersion on impact, as new adjusters are *selected* from those with the most misaligned markups; while aggregate shocks increase markup dispersion in the Calvo framework. Furthermore, resource costs of price changes become a relevant factor in our framework, while they are always zero in the Calvo framework by construction. The policymaker's task is to minimize the effect of those distortions.

We find that optimal monetary policy should lean more aggressively against inflation after large cost-push shocks when the frequency of price changes is endogenously high than either after a small shock or in a fixed-frequency Calvo setting: it is optimal to "strike while the iron is hot".[7] This nonlinearity establishes a key difference between our model and the standard Calvo model. Our calibration implies that this new policy prescription is relevant for the 2022-2023 inflation surge: Already for inflation and frequency values of the magnitude observed during this period, optimal policy requires a significantly more aggressive anti-inflationary stance than for a small shock or under Calvo pricing.

What explains the modified policy prescription? To gain insight into this result, we introduce a simplified model. In the simplified model, we introduce a sub-period

[4]See, for example, Nakamura and Steinsson (2008).

[5]The Calvo model is recalibrated to generate the same price-flexibility as our baseline model for small shocks (Auclert et al. 2024). This recalibration compensates for the endogenous "selection" of large price changes, which substantially raises the flexibility of the aggregate price level.

[6]As is standard in optimal monetary policy analysis, we offset steady-state average markup distortion due to the market power with suitable subsidies. We reintroduce steady-state markup distortions only to analyze the impact of time-inconsistency.

[7]We analyze timeless Ramsey policy á la Woodford (2003).

of night, when only the firms are awake, and when the prices are fully flexible. The assumption improves the tractability of the model by turning the dynamic problem of the firms into a series of static problems, but it keeps the key underlying channel active: the repricing rate responds endogenously to aggregate shocks.

In the simplified model, both welfare and the planner's choice set can be expressed in the space of (i) the output gap, which measures the distance between output and its efficient level, and (ii) inflation, as is conventional in optimal monetary policy analysis under the Calvo price setting. Welfare depends on these two variables because the output gap is related to the average markup, and inflation is related to the markup dispersion and the resource costs of price changes. The choice set can also be expressed in the space of inflation and output gap – and takes the form of a nonlinear Phillips curve. It is nonlinear because inflation becomes more sensitive to changes in the output gap as large shocks raise the frequency of price changes and, thereby, increase price flexibility.[8]

Optimal policy leans more aggressively against inflation after large cost-push shocks than after small shocks in the simplified model, just as in the full model. In the simplified model, the relationship between the output gap and inflation under optimal policy can be illustrated by a structural "target rule." The optimal "strike while the iron is hot" policy translates into a nonlinear target rule: larger output gaps are associated with *relatively* lower inflation rates than smaller output gaps. This is in stark contrast to the corresponding target rule in the Calvo framework, which is almost linear.

To understand the key driving forces behind this policy prescription, it is instructive to start with the question of why the target rule is almost linear in the Calvo model. There, welfare can be well approximated by a quadratic function of the output gap and inflation with a fixed weight[9], while the Phillips curve is almost linear. Optimal policy thus maximizes a near-quadratic objective subject to the near-linear Phillips curve. The resulting policy is thus near linear.

Why is the same relationship between inflation and output gap nonlinear under state-dependent pricing? We find that this is almost exclusively driven by the nonlinear trade-off between inflation and output gap – the nonlinear Phillips curve. Intuitively, reducing inflation in this framework is cheaper after large shocks, when the frequency is higher and the price level is more flexible, that is, when the sacrifice ratio is low. To show that this is the dominant driving force, we combine the nonlinear Phillips curve of our simplified framework with a counterfactual quadratic welfare function approximating the Calvo model and derive a counterfactual target rule. We show that the ensuing target rule is close to the true target rule and is similarly characterized by the "strike while the iron is hot" policy. As the relative welfare weight of inflation

[8] Other papers point to complementary reasons why the Phillips curve can be nonlinear, such as state-dependent wage rigidity (Benigno and Eggertsson 2023) or Kimball aggregators (Erceg et al. 2024).

[9] Woodford (2003) shows that welfare can be approximated by a weighted sum of squares of output gap and inflation, the weight being determined by structural parameters of the model. This approximation applies in the neighborhood of an efficient steady state up to a second order, i.e. for small shocks. We find that this approximation works well numerically also for large shocks.

and output gap is independent of the shock in this counterfactual by construction, the results here are clearly driven by the shape of the Phillips curve – the nonlinear sacrifice ratio.[10] This result generalizes to the full model. Therefore, we conclude that the key driving force behind the aggressive anti-inflationary stance after large shocks is the lower sacrifice ratio.

We establish a series of additional results in the full model. First, we show that the model features a slightly positive Ramsey optimal steady-state inflation rate, at around 0.07% per annum. This contrasts with the standard Calvo model, where optimal inflation is exactly zero. In our menu cost model, slightly positive steady state inflation reduces the frequency and thus helps firms to economize on costly price adjustments. In particular, it counterbalances the impact of too frequent price increases relative to price decreases, which is a consequence of the asymmetry of the profit function: firms dislike more negative price misalignments when the demand for their product is high, relative to positive misalignments when the demand is low. Second, we also find that for small cost-push shocks, optimal policy "leans against the wind": the central bank temporarily drives output below its efficient level to contain the inflationary impact of a positive cost-push shock. This is very similar, though not identical, to the Calvo model. However, the reason is different. In the Calvo model, the key distortion caused by inflation is the markup dispersion, while in our baseline model it is the resource costs due to menu costs. Third, we show analytically that the optimal response to TFP shocks is characterized by the "divine coincidence" (Blanchard and Galí 2007). In other words, optimal policy stabilizes both inflation and the output gap. Finally, we show that the well-known time inconsistency problem of optimal monetary policy is also present in our menu cost model, although it is attenuated relative to Calvo. In both models, when the steady state is inefficient, monetary policy has the incentive to stimulate output via an unexpectedly easy policy (Galí 2008). However, in the menu cost model, such a policy is less effective on output and more inflationary because the ensuing increase in the repricing rate raises the flexibility of the aggregate price level. The time-inconsistent motive to ease is thus considerably weaker.

Our results are robust to alternative parameterizations, and also hold in the "Calvo-Plus" model (Nakamura and Steinsson 2010). The latter framework assumes that the price adjustment cost is stochastic: it takes a positive value with some exogenous probability, and it is zero otherwise. This model can better match the fraction of small price changes in the data (see also Midrigan 2011; Alvarez et al. 2021), and can achieve a realistic degree of monetary non-neutrality for small shocks. However, as we show, the model also prescribes a more aggressive anti-inflationary stance after large cost-push shocks than after small shocks, with an even higher nonlinearity than our baseline. This is primarily because the sacrifice ratio exhibits even more nonlinearity in this framework than in our baseline.

[10] The deviation of the true welfare from the quadratic approximation actually somewhat *mitigates* the nonlinearity of the true target rule, but its impact is quantitatively small.

Related literature. Our paper builds on the seminal article by Golosov and Lucas (2007). They propose a menu cost model (Barro 1972; Sheshinski and Weiss 1977; Caballero and Engel 1993) that provides a micro-founded state-dependent alternative to the canonical time-dependent Calvo (1983) model. The framework has become the backbone of a positive literature (Gertler and Leahy 2008; Midrigan 2011; Costain and Nakov 2011; Alvarez et al. 2016; Auclert et al. 2024)[11] and is shown to describe firms' price-setting behavior well in diverse environments with both low and high inflation (Nakamura and Steinsson 2008; Gagnon 2009; Alvarez et al. 2019; Nakamura et al. 2018), and as response to large aggregate shocks (Karadi and Reiff 2019; Alexandrov 2020; Auer et al. 2021).

To the best of our knowledge, our paper is the first to solve for optimal monetary policy in this canonical menu cost model. Its main distinctive feature is the endogeneity of the extent of price stickiness: the frequency of price changes can vary with macroeconomic conditions and thus it is endogenous to shocks and potentially to monetary policy itself. This is different from the canonical textbook analysis of optimal monetary policy based on Calvo (1983), such as in Woodford (2003) and Galí (2008). Changes in frequency have been documented both after large aggregate shocks (Karadi and Reiff 2019; Alvarez and Neumeyer 2020; Auer et al. 2021; Gagliardone et al. 2025; Gautier et al. 2025), and in high-inflation environments (Gagnon 2009; Alvarez et al. 2019; Nakamura and Steinsson 2018), and have received new empirical support following the recent U.S. inflation surge (Montag and Villar 2023; Cavallo et al. 2024; Blanco et al. 2024a). Variation in frequency implies a state-dependent, nonlinear relationship between inflation and the output gap (Vavra 2014; Blanco et al. 2024b). Our conclusion prescribing an aggressive anti-inflationary policy after large shocks is a direct consequence of this nonlinearity, which implies a favorable inflation-output trade-off that optimal policy should exploit.

Solving dynamic optimal policy in response to aggregate shocks in this framework complements previous research on optimal monetary policy, which has restricted its attention to menu cost settings with a representative firm and small aggregate shocks (Nakov and Thomas 2014), sector-specific productivity shocks (Caratelli and Halperin 2023) or to optimal *steady-state* inflation (Adam and Weber 2019; Blanco 2021; Nakov and Thomas 2014).[12]

This paper proposes a new algorithm to solve Ramsey optimal policy in heterogeneous-agent models, building on González et al. (2024). The algorithm (i) makes the infinite-dimensional planner's problem finite-dimensional by approximat-

[11] A key question of the literature is the relationship between monetary non-neutrality and the distribution of price changes at the micro level. This is not the focus of our paper. We show that our results on optimal monetary policy after large shocks are robust across models with very different implications about monetary non-neutrality, like the Golosov and Lucas (2007) model on the one hand and the CalvoPlus model of Nakamura and Steinsson (2010) on the other.

[12] Nakov and Thomas (2014) find no significant difference between Calvo and a random menu cost model. Caratelli and Halperin (2023) show that, in the face of sector-specific shocks, optimal policy can be characterized as *nominal wage targeting*.

ing the infinite-dimensional value and distribution functions by piece-wise linear functions on a grid; (ii) accounts for the discrete price-adjustment choice using an endogenous grid; (iii) derives the FOCs of the planner's problem by symbolic differentiation; and (iv) solves the resulting set of equilibrium conditions nonlinearly under perfect foresight over the sequence space. Our approach complements other methods to solve for Ramsey policy in heterogeneous-agent models (Bhandari et al. 2021; Le Grand et al. 2022; Dávila and Schaab 2022; Nuño and Thomas 2022; Smirnov 2022).

2. Model

In the baseline economy a representative household consumes a basket of differentiated goods and supplies labor; monopolistic firms produce using a technology that is affected by both aggregate and idiosyncratic shocks and must pay a fixed menu cost to change prices; and a central bank sets interest rates. Time is discrete and there is no aggregate uncertainty. We compare our baseline economy to a Calvo economy, which is identical to our baseline except firms adjust their prices with an exogenous probability.

2.1. Households

A representative household consumes C_t, supplies working hours N_t and saves in one-period, nominal bonds B_t, which are in zero net supply. The household maximizes

$$\max_{C_t,N_t,B_t} \sum_{t=0}^{\infty} \beta^t u(C_t, N_t), \tag{1}$$

subject to

$$P_tC_t + Q_tB_t + T_t \quad = \quad B_{t-1} + W_tN_t + D_t, \tag{2}$$

where T_t are lump-sum taxes, W_t is the nominal wage, D_t are lump-sum dividends from firms, and $Q_t \equiv e^{-i_t}$ is the price of the nominal bond and i_t is the nominal interest rate. Aggregate consumption C_t is

$$C_t = \left\{ \int_0^1 \left[A_t(j)C_t(j)\right]^{\frac{\epsilon-1}{\epsilon}} dj \right\}^{\frac{\epsilon}{\epsilon-1}}, \tag{3}$$

where $C_t(j)$ is the quantity of product $j \in [0, 1]$ and $A_t(j)$ is the idiosyncratic quality of product j, which follows a random walk in logs with volatility σ:

$$\log A_t\,(j) = \log A_{t-1}\,(j) + \sigma\varepsilon_t\,(j)\,, \tag{4}$$

ε_t is an i.i.d Gaussian innovation and σ is a parameter.

The demand for product j is

$$C_t(j) = A_t(j)^{\epsilon-1}\left(\frac{P_t(j)}{P_t}\right)^{-\epsilon} C_t \tag{5}$$

where $P_t(j)$ is the price of product j, and the aggregate price index is

$$P_t = \left[\int_0^1 \left(\frac{P_t(j)}{A_t(j)}\right)^{1-\epsilon} dj\right]^{\frac{1}{1-\epsilon}}. \tag{6}$$

We assume separable utility $u(C_t, N_t) = \log C_t - N_t$ as in Midrigan (2011). Thus, equilibrium in the labor market requires:

$$w_t = C_t, \tag{7}$$

where $w_t = W_t/P_t$ is the real wage. The Euler equation is

$$1 = \left[\Lambda_{t,t+1} e^{i_t - \pi_{t+1}}\right], \tag{8}$$

where i_t is the nominal interest rate, and the the stochastic discount factor is

$$\Lambda_{t,t+1} \equiv \beta \frac{u'(C_{t+1})}{u'(C_t)}. \tag{9}$$

2.2. Monopolistic producers

Good $j \in [0,1]$ is produced by firm j according to a constant-returns to scale technology

$$Y_t(j) = A_t \frac{N_t(j)}{A_t(j)}, \tag{10}$$

where $N_t(j)$ is labor hours, A_t is aggregate productivity and $A_t(j)$ is idiosyncratic quality. Firms maximize the sum of the discounted future profits, using the household's discount factor. They take the demand function (5) as given. Firm j's nominal profit function given its nominal price $P_t(j)$ is

$$\begin{aligned} D_t(j) =& P_t(j)Y_t(j) - (1-\tau_t)W_t N_t(j) \\ =& P_t(j)^{1-\epsilon} A_t(j)^{\epsilon-1}\left(\frac{1}{P_t}\right)^{-\epsilon} C_t - (1-\tau_t)\frac{W_t}{A_t} A_t(j)^{\epsilon}\left(\frac{P_t(j)}{P_t}\right)^{-\epsilon} C_t \end{aligned} \tag{11}$$

where τ_t is an employment subsidy financed by lump-sum taxes, and where for the second line we have used the goods market-clearing condition $Y_t(j) = C_t(j)$ and conditions (5) and (10).

Crucially, firm j must incur a fixed "menu cost" η in labor units to change its price. The firm chooses each period whether to update its nominal price to a new one $P_t^*(j)$, or to keep the price from last period $P_{t-1}(j)$. This is the source of endogeneity of price stickiness in the model.

It is useful to express the firms' optimal pricing decision as a function of the *price gap* $x_t(j) \equiv p_t(j) - p_t^*(j)$, which is the log distance between the current, $p_t(j) \equiv \log\left(\frac{P_t(j)}{A_t(j)P_t}\right)$, and the optimal quality-adjusted relative price of good j, $p_t^*(j) \equiv \log\left(\frac{P_t^*(j)}{A_t(j)P_t}\right)$. Thanks to the convenient assumptions of random-walk quality shocks as in Midrigan (2011), the price gap is the only firm-level state variable that the pricing decision depends upon. The firms' optimal pricing policy follows a Ss rule such that a firm j keeps its nominal price $P_t(j)$ constant if $x_t(j) \in [s_t, S_t]$, and resets it to the optimal price $P_t^*(j)$ otherwise (equivalent to set $x_t(j) = 0$). Subindex t subsumes all aggregate states. Idiosyncratic quality shocks generates heterogeneity across nominal reset prices $P_t^*(j)$; however, quality-adjusted relative reset prices $p_t^*(j)$ are all identical. Thus, we drop subindex j to simplify notation. When a firm keeps its nominal price constant, its price gap evolves according to

$$x_t = x_{t-1} + p_t - p_t^* + p_{t-1}^* = x_{t-1} - \sigma\varepsilon_t - \pi_t^* \tag{12}$$

where

$$\pi_t^* \equiv p_t^* - p_{t-1}^* + \pi_t \tag{13}$$

is the inflation of the quality-adjusted relative optimal reset price.

The optimality conditions for the pricing rule require

$$V_t'(0) = 0, \tag{14}$$

$$V_t(0) - \eta w_t = V_t(s_t), \tag{15}$$

$$V_t(0) - \eta w_t = V_t(S_t), \tag{16}$$

where firms' end of period value function $V_t(\cdot)$ is expressed only in terms of price gaps as all other states are aggregate and are subsumed by a time subindex. Equation (14) requires that the value function is maximized at the optimal reset price ($x = 0$), and equations (15, 16) require indifference between resetting the price and paying the menu cost versus keeping prices constant at the endogenous Ss thresholds. The value function equals

$$\begin{aligned} V_t(x) =&\Pi_t(x) + \frac{\Lambda_{t,t+1}}{\sigma}\int_{s_{t+1}}^{S_{t+1}}\left[V_{t+1}(x')\phi\left(\frac{x - x' - \pi_{t+1}^*}{\sigma}\right)\right]dx' \\ &+ \Lambda_{t,t+1}\left(1 - \frac{1}{\sigma}\int_{s_{t+1}}^{S_{t+1}}\left[\phi\left(\frac{x - x' - \pi_{t+1}^*}{\sigma}\right)\right]dx'\right)\left[\left(V_{t+1}(0) - \eta w_{t+1}\right)\right], \end{aligned} \tag{17}$$

where the current real profits $\Pi_t(x)$ are given by

$$\Pi_t(x) \equiv \frac{D_t}{P_t} = C_t e^{(x+p_t^*)(1-\epsilon)} - C_t(1-\tau_t)\frac{w_t}{A_t}e^{(x+p_t^*)(-\epsilon)}. \tag{18}$$

The value function is the sum of current profits and the discounted continuation value. The latter depends on the firms' price gap next period, which, unless changed, evolves according to (12) and is affected by the stochastic component ε whose density is $\phi(\varepsilon)$. The continuation value then consists of two parts. The first measures the expected value $V_{t+1}(x')$ in the states of the world where the price is not changed, i.e. when x' remains within the inaction threshold $[s_{t+1}, S_{t+1}]$. The second measures the expected value for the states of the world where the price is updated to $x' = 0$, which is given by $V_{t+1}(0)$ net of menu cost ηw_{t+1}.

Finally, Appendix D shows that $V_t'(0)$ can be expressed as the sum of the marginal effect of x on current profits and on the expected continuation value:

$$V_t'(0) = \Pi_t'(0) + \frac{\Lambda_{t,t+1}}{\sigma} \int_{s_{t+1}}^{S_{t+1}} V_{t+1}(x') \left. \frac{\partial \phi\left(\frac{x - x' - \pi_{t+1}^*}{\sigma}\right)}{\partial x} \right|_{x=0} dx'$$
$$+ \frac{\Lambda_{t,t+1}}{\sigma} \left[\phi\left(\frac{-S_{t+1} - \pi_{t+1}^*}{\sigma}\right) - \phi\left(\frac{-s_{t+1} - \pi_{t+1}^*}{\sigma}\right) \right] \left(V_{t+1}(0) - \eta w_{t+1}\right).$$

2.3. Aggregation and equilibrium conditions

Firms' individual price-setting decisions give rise to an endogenous probability density of end-of-period price gaps $g_t(x)$. It consists of a continuous part, $g_t^c(x)$, and a mass point (dirac delta) at $x = 0$, g_t^0 such that

$$g_t(x) \equiv g_t^c(x) + g_t^0 \delta(x). \tag{19}$$

The countinous part evolves according to the following law of motion

$$g_t^c(x) = \begin{cases} \frac{1}{\sigma} \int_{s_{t-1}}^{S_{t-1}} g_{t-1}^c(x_{-1}) \phi\left(\frac{x_{-1} - x - \pi_t^*}{\sigma}\right) dx_{-1} + g_{t-1}^0 \phi\left(\frac{-x - \pi_t^*}{\sigma}\right), & \text{if } x \in [s_t, S_t], \\ 0, & \text{otherwise,} \end{cases} \tag{20}$$

and the mass point evolves according to

$$g_t^0 = 1 - \int_{s_t}^{S_t} g_t^c(x) dx. \tag{21}$$

The first term on the right-hand side in the first line of equation (20) describes the evolution of the density of price gaps of those firms that kept their nominal prices unchanged last period, while the second term is the distribution of current price gaps of the firms that did change their prices last period. Outside the Ss band, there is no mass, since firms whose price would fall outside the Ss band reset their prices in the current period, thus creating a mass point at zero, equal to the frequency of price changes (equation 21).

The aggregate price index implies

$$1 = \int_{s_t}^{S_t} e^{\left(x+p_t^*\right)(1-\epsilon)} g_t\left(x\right) dx. \tag{22}$$

In turn, the labor-market clearing condition is given by

$$N_t = \underbrace{\frac{C_t}{A_t} \int_{s_t}^{S_t} e^{-\epsilon\left(x+p_t^*\right)} g_t}_{\text{price dispersion}} \quad + \quad \underbrace{\eta g_t^0}_{\text{price adjustment cost}} \tag{23}$$

such that the total hours worked equals the total use of labor for the production of aggregate output $\left(\text{the term } \frac{C_t}{A_t}\right)$ adjusted for the loss in aggregate output due to price dispersion (the term in parenthesis) and the total amount of hours allocated to price adjustment (the second term in the right-hand side).

The 12 equations (7), (8), (13) - (17), (19)- (23) plus $C_t = Y_t$ define the private equilibrium in $w_t, N_t, Y_t, C_t, V_t(\cdot), S_t, s_t, p_t^*, \pi_t, \pi_t^*, g_t(\cdot), g_t^c(\cdot), g_t^0, i_t$. The central bank has one degree of freedom to set the nominal rate.

2.4. Aggregate Shocks

The logarithm of aggregate productivity follows a first-order autoregressive process:

$$\log A_t = \rho_A \log A_{t-1} + \varepsilon_{A,t}, \tag{24}$$

where $\rho_A \in [0,1]$ and $\varepsilon_{A,t}$ is an aggregate productivity shock, which arrives unexpectedly. The employment subsidy τ_t follows the autoregressive process:

$$\tau_t - \tau = \rho_\tau(\tau_{t-1} - \tau) + \varepsilon_{\tau,t}, \tag{25}$$

where $\rho_\tau \in [0,1]$, τ is the steady-state employment subsidy, and $\varepsilon_{\tau,t}$ is an unexpected cost-push shock.

2.5. Auxiliary models

We briefly present three alternative models we use in our analysis.

Simplified model. We also present a simplified version of the model to foster intuition. The simplification consists of dividing each period t into two: a night and a day. The day is as in the full model. What's new is the night, when only firms are awake and can reset their prices for free.

Under this setup, first, the price gap distribution collapses to a mass point at $x = 0$ every night by construction as each firm closes its price gap. Second, when making a decision during the day, firms set prices only with the current period in mind, as if their discount rate were zero ($\beta = 0$). They do this because they know that they will be able to reset their prices for free again the next night.

The simplification maintains the state-dependent nature of the firms' price-setting problem: both how many and which prices change will be decided endogenously as a response to the aggregate and idiosyncratic shocks, taking into account the conduct of monetary policy. We gain tractability, however, by replacing the dynamic problem of the firms with a series of static problems, where future expectations play no role in the price-setting decisions. The advantage of the approach is that, as we show below, objects familiar from conventional optimal policy analysis like the Phillips curve, which describes the trade-off between inflation and output gap, and the target rule, which describes the relationship between inflation and output gap under optimal policy, become structural. Despite its simplicity, the model generates results that are not only qualitatively but also quantitatively similar to analogous objects in the full model.

Thanks to this simplifying assumption, the firm's value function (17) collapses to the current profit function (18). Therefore the optimality condition for the reset price (14) simplifies to a constant markup over marginal costs. Dropping the time index t for brevity, it reads

$$e^{p^*} = \frac{\epsilon}{(\epsilon - 1)}(1-\tau)w. \tag{26}$$

The firms' price adjustment thresholds (15) and (16) now characterize the threshold values which equate current profits under unchanged prices, $\Pi(x)$, with profits under the optimal price net of the menu costs, $\Pi(0) - \eta w$

$$\begin{aligned}
(e^{p^*})^{1-\epsilon} - (1-\tau)w(e^{p^*})^{-\epsilon} - \eta &= e^{(p^*+s)(1-\epsilon)} - (1-\tau)we^{(p^*+s)(-\epsilon)} &\quad (27)\\
(e^{p^*})^{1-\epsilon} - (1-\tau)w(e^{p^*})^{-\epsilon} - \eta &= e^{(p^*+S)(1-\epsilon)} - (1-\tau)we^{(p^*+S)(-\epsilon)}. &\quad (28)
\end{aligned}$$

Free price changes in the preceding night implies that $g^c_{-1}(x) = 0$, $g^0_{-1} = 1$, $p^*_{-1} = 0$ and, therefore, the price gap distribution g^c is now normally distributed with its mean given by $(\pi + p^*)$ and with variance σ^2:

$$g^c(x) = \frac{1}{\sigma}\phi\left(\frac{x + \pi + p^*}{\sigma}\right) \text{ if } x \in [s, S]. \tag{29}$$

These 4 equations, together with 4 equations which don't change relative to the full model (labor supply (7), frequency of price changes (21), labor-market-clearing (23), definition of the price level (22)) define an equilibrium in 9 variables $w, \pi, C, N, s, S, g^0, g^c(\cdot), p^*$. The policymaker has one degree of freedom to choose π.[13]

Nonlinear Calvo model. This model is a natural benchmark that we contrast our model to. It is identical to the baseline economy without idiosyncratic shocks $\sigma = 0$ or menu

[13]To ensure that firms have no incentive to deviate from a symmetric reset price at night, we assume that the value of τ expected for the next day is such that $\pi = 0$ is the optimal policy.

costs $\eta = 0$. Instead, firms face an exogenous price change probability θ as in Calvo (1983). We will use this benchmark both in the full and in the simplified setup.

CalvoPlus model. We use this model to study the robustness of our main results. It is an extension of our baseline model proposed by Nakamura and Steinsson (2010), where the menu cost is stochastic: it equals η with probability α and zero otherwise. The extension improves the realism of the framework both by better capturing the distribution of price changes through the introduction of small price changes, and better matching the extent of monetary non-neutrality obtained by time-series evidence. Appendix G describes the details.

3. Optimal monetary policy problem and computational approach

We start our analysis by introducing the central bank's problem. We consider the Ramsey problem, i.e., optimal monetary policy under commitment. We also present a new computational method to deal with the complexities associated with the problem's high dimensionality, and we specify our baseline calibration.

3.1. Ramsey problem

The central bank selects the paths for all equilibrium variables subject to the competitive equilibrium conditions. Combining households' utility function in (1) and the market-clearing conditions for final output $C_t = Y_t$ and for labor (23), the problem of a benevolent central bank is

$$\max_{\substack{\{w_t, Y_t, V_t(\cdot), S_t, s_t, \\ p_t^*, \pi_t, \pi_t^*, g_t(\cdot), g_t^c(\cdot), g_t^0\}_{t=0}^{\infty}}} \sum_{t=0}^{\infty} \beta^t \left(\log Y_t - \frac{Y_t}{A_t} \int_{s_t}^{S_t} e^{-\epsilon(x+p_t^*)} g_t(x)\, dx - \eta g_t^0 \right)$$

subject to the labor supply (7), firms' value function $V_t(\cdot)$ (17), firms' optimal pricing $\{s_t, S_t, p_t^*\}$ (14), (15), and (16), the definition for inflation in quality-adjusted relative optimal reset price π_t^* (13), the distribution of price gaps $(g_t(\cdot), g_t^c(\cdot), g_t^0)$ determined by equations (19)-(21), and the aggregate price index (22).

Two observations are due. First, we follow the approach in standard optimal monetary analysis (Woodford 2003; Galí 2008) of separating the Ramsey problem in two: the equilibrium pinned down by a benevolent central bank and the implementation problem, i.e. the nominal interest rates path consistent with the equilibrium according to the household's Euler equation (8).

Second, note that the constraints set for this problem are continuous and differentiable even though the individual firm's price policy function is not. This is so because each firm has zero mass, and thus the discontinuity in a single firm's behavior does not lead to a discontinuity in aggregates. Furthermore, $V_t(x)$ and $g_t^c(x)$ are continuously differentiable over the relevant range (s_t, S_t).

3.2. Computational solution method

We solve the problem with a new nonlinear algorithm, which extends the approach in González et al. (2024) to discrete time. The core idea is to represent the Ramsey problem of the central bank as a high-dimensional optimization problem in which the Bellman equation and the law of motion (LOM) of the price-gap distribution are constraints. We summarize the approach here, while Appendix F presents the details.

The solution of this Ramsey problem poses several challenges. First, the value function $V_t(\cdot)$ and the distribution $g_t^c(\cdot)$ are infinite-dimensional variables and we need to compute the first-order conditions (FOCs) with respect to these variables.[14] Second, any approximation of the problem needs to account for the discrete choices of the firm and to be smooth and accurate enough to capture the higher-order effects of policy.

The first step consists in transforming the original infinite-dimensional problem into a high-dimensional problem by discretizing the value and distribution functions. To this end, we replace the distribution and value functions by piecewise linear functions over a grid. The grid itself is endogenous. It is selected to include the two bounds of the inaction region $[s_t, S_t]$ and the optimal price ($x_t = 0$) at each t.

Next, integrals to compute expectations are evaluated algebraically, conditional on those piecewise linear functions. Then, we require the law of motion of the distribution and the Bellman equation to hold exactly at the (endogenous) grid points at each period t. In doing so, we explicitly take the mass point at 0 into account in the distribution, in line with the notation in the paper. This transforms the private equilibrium conditions into a large system of difference equations. In particular, the firms' Bellman equation at time t can be approximated over a grid of price gaps x as

$$\mathbf{V}_t = \mathbf{\Pi}_t + \left[\mathbf{A}_t \mathbf{V}_{t+1} - \mathbf{b}_{\mathbf{t+1}} \eta w_{t+1}\right],$$

where $\mathbf{V}_t$ and $\mathbf{b}_t$ are vectors of the value function and the expected adjustment probability evaluated at different grid points, respectively, and $\mathbf{A}_t$ is a matrix that captures the idiosyncratic transitions due to firm-level quality shocks and aggregate inflation.

[14] There are a number of proposals in the literature to deal with this problem. Nuño and Thomas (2022), Smirnov (2022), and Dávila and Schaab (2022) deal with the full infinite-dimensional planner's problem in continuous time. This implies that the Kolmogorov forward (KF) and the Hamilton-Jacobi-Bellman (HJB) equations are constraints faced by the central bank. They derive the planner's FOCs using calculus of variations, thus expanding the original problem to also include the Lagrange multipliers, which in this case are also infinite-dimensional. These papers solve the resulting differential equation system using the upwind finite-difference method of Achdou et al. (2021). Bhandari et al. (2021) make the continuous cross-sectional distribution finite-dimensional by assuming that there are N agents instead of a continuum. They then derive standard FOCs for the planner. In order to cope with the large dimensionality of their problem, they employ a perturbation technique. Le Grand et al. (2022) employ the finite-memory algorithm proposed by Le Grand and Ragot (2022). It requires changing the original problem such that, after K periods, the state of each agent is reset. In this way the cross-sectional distribution becomes finite-dimensional.

Similarly, the law of motion of the density for $x \neq 0$ is

$$\mathbf{g}_{\boldsymbol{t}}^{\boldsymbol{c}} = \mathbf{F}_t \mathbf{g}_{\boldsymbol{t-1}}^{\boldsymbol{c}} + \mathbf{f}_t g_{t-1}^0,$$

where $\mathbf{g}_{\boldsymbol{t}}^{\boldsymbol{c}}$ and $\mathbf{f}_t$ are vectors representing the probability distribution function and the scaled and shifted normal distribution, respectively, $\mathbf{F}_t$ is a matrix that captures the evolution of the price distribution due to firm-level quality shocks and aggregate inflation. We define

$$g_t^0 = 1 - \mathbf{e}_t^\top \mathbf{g}_{\boldsymbol{t}}^{\boldsymbol{c}}.$$

as the mass point at $x_t = 0$ where $\mathbf{e}_t$ is a vector of weights corresponding to the trapezoid rule. The labor market clearing condition and the aggregate price index can be written in a similar form.

Once we have the discretized version of the problem, we find the Ramsey planner's FOCs by symbolic differentiation. We are now left with an even larger system of difference equations, as we have FOCs for the value and distribution functions at each grid point, and the associated Lagrange multipliers.

Next, we find the Ramsey steady state. To do so, we use the steady-state private equilibrium conditions to construct a nonlinear multidimensional function mapping inflation to the rest of the variables. We then combine this function with the planner's FOCs. As this system is linear in Lagrange multipliers, finding its solution boils down to finding the root of a nonlinear uni-variate function in inflation. To do so we use the Newton method. Finally, to compute the dynamics of the Ramsey problem, we solve the system of difference equations non-linearly in the sequence space, also using the Newton method.

The symbolic differentiation and the two applications of the Newton method can be automated using several available software packages, in our case, Dynare (Adjemian et al. 2023). The approach is also compatible with the *nonlinear* sequence-space Jacobian toolbox by Auclert et al. (2021). It can be employed to compute optimal policies in a large class of heterogeneous-agent models. Compared to other algorithms, it stands out as easy to implement. In our application, it runs in a few minutes on a normal laptop. As González et al. (2024) show, this algorithm delivers the same results as computing the planner's FOCs using calculus of variations and then discretizing the resulting system of differential equations.

3.3. Calibration

Table 1 presents the calibration of our baseline and the simplified model. One period is one months. We set the discount factor to $0.96^{1/12}$, which implies a steady-state real interest rate of 4%. The elasticity of substitution across products is $\epsilon = 7$, as in Golosov and Lucas (2007).

We calibrate the menu cost and the standard deviation of idiosyncratic shocks to match two target moments: an 8.7% monthly frequency of price changes in the

	β	ϵ	η	σ	τ	ρ_A	ρ_τ
Baseline	$0.96^{1/12}$	7	0.010	0.012	0.1435	$0.95^{1/3}$	$0.9^{1/3}$
Simplified model	$0.96^{1/12}$	7	0.004	0.021	0.1431	0	0

TABLE 1. Parameter values

steady state as documented for the U.S. in Nakamura and Steinsson (2008), and a 20% frequency at 10% inflation rate broadly in line with the peak values observed in the U.S. in 2022 as documented by Montag and Villar (2023). In the baseline model, the implied menu cost is $\eta = 1\%$ and the steady-state standard deviation of idiosyncratic quality shocks is $\sigma = 1.2\%$.[15] The steady-state labor subsidy τ is set to ensure that output is at its efficient level.

Finally, the persistence of shocks is taken from Smets and Wouters (2007), once transformed from quarterly to monthly frequency: $\rho_A = 0.95^{1/3}$ for aggregate productivity shocks and $\rho_\tau = 0.9^{1/3}$ for employment subsidy shocks (interpreted as cost-push shocks).

These parameters are inherited also by the additional auxiliary models. The same calibration targets imply an $\eta = 0.4\%$ and $\sigma = 2.1\%$ in the simplified model. In the Calvo model, we disregard idiosyncratic shocks $\sigma = 0$ and calibrate the probability of price adjustment $(1-\theta)$ so as to make the Calvo model imply an identical response to a small monetary policy shock as our baseline model (Auclert et al. 2024). This requires a parameter $\theta = 40\%$.

4. Strike while the iron is hot

This section focuses on our main result: the nonlinearity of optimal monetary policy in response to cost-push shocks.[16] We first present numerical simulations in our full model to characterize the nature of the nonlinearity and to contrast it with the conventional Calvo framework. Then we describe its main driving forces relying on the simplified model. We close the section by showing robustness in the CalvoPlus model.

The section analyzes *timeless* optimal monetary policy (Woodford 2003; Galí 2008). This corresponds to the optimal monetary policy starting from the Ramsey steady state, when all of the Lagrange multipliers are initialized at their steady-state values.[17]

[15] The calibration does not match the average absolute size of price changes. Our results would be qualitatively similar under the alternative calibration that would match the steady state frequency and the size of price changes (not shown).

[16] We defer the analysis of the steady state of the optimal Ramsey problem to Section 5.

[17] We assess the implications on the time inconsistency of optimal policy in our state-dependent framework in Section 5. As long as the labor subsidy τ in the steady state offsets the average markup distortion, as in our baseline calibration, the optimal policy is virtually time consistent.

4.1. Nonlinear optimal monetary response to cost-push shocks

How should optimal monetary policy react to cost-push shocks of different sizes, and how do reactions in our baseline model compare to those in the canonical Calvo framework? Figure 1 shows impulse responses to a large cost-push shock ($\varepsilon_{\tau,t}$, blue solid line) in the baseline model, and contrasts it to linearly-scaled impulse responses to a small cost-push shock (yellow dotted line); and to a large cost-push shock in the Calvo model (red dashed line).[18] The size of the large shock is calibrated to generate a 20% frequency at the peak in the baseline model, a 12.3 percentage point increase from the 8.7% frequency at the steady state. The magnitude of the frequency increase is broadly in line with that observed during the 2022-2023 inflation surge in the U.S. (Montag and Villar 2023).[19]

The optimal policy response "leans against the wind" in all three cases. The central bank tolerates an inflation increase (panel a) to partially cushion the decline in output (panel b).[20] Optimal policy implies a temporary decline in the real interest rate in parallel with the spiking inflation, but prescribes a commitment to a persistently tight policy stance in the future.

Optimal monetary policy in the menu cost model is nonlinear. Impulse responses to the large shock are significantly different from the linearly-scaled responses to a small shock. Notably, the frequency under a large shock increases substantially, while it remains almost unchanged after the small shock, even though it is linearly scaled (panel d). This nonlinear frequency response is an inherent feature of the model. Consider a small inflationary shock. The repricing frequency stays unchanged because the fall in the frequency of price decreases almost completely offsets the rise in the frequency of price increases. If one considers instead a large inflationary shock, the price decreases fade out and the rise in the frequency of price increases quickly dominates, thus producing an overall increase in the repricing frequency (Gagnon 2009; Karadi and Reiff 2019; Alvarez and Neumeyer 2020; Alexandrov 2020; Cavallo et al. 2024).

The optimal policy is characterized by a more aggressive monetary policy (panel c) after the large cost-push shock, which raises the frequency of price changes, than after the small shock. The central bank "strikes while the iron is hot." The tighter policy leads to a substantially more muted increase in inflation after the large shock than after the linearly-scaled small shock (panel a). The output effects are broadly similar (panel b).[21]

[18]The scaling factor is the ratio between the shock size of the large (68%) and the small shock (0.25%).

[19]The impulse responses are computed nonlinearly under perfect foresight. For small shocks, this is equivalent to the first-order approximation to the stochastic problem, as discussed by Boppart et al. (2018). For large shocks, its interpretation is similar to that in Cavallo et al. (2024): an unexpected once-and-for-all large shock that hits the economy in the deterministic steady state.

[20]For cost-push shocks, output equals the output gap, as this type of shock yields no variation of efficient output.

[21]The output effect is somewhat smaller for large adverse cost-push shocks than for linearly-scaled small shocks in the Calvo model (see Figure 2). This nonlinearity of the underlying framework is inherited in the menu cost model, which explains why output declines slightly more in the case of small shocks.

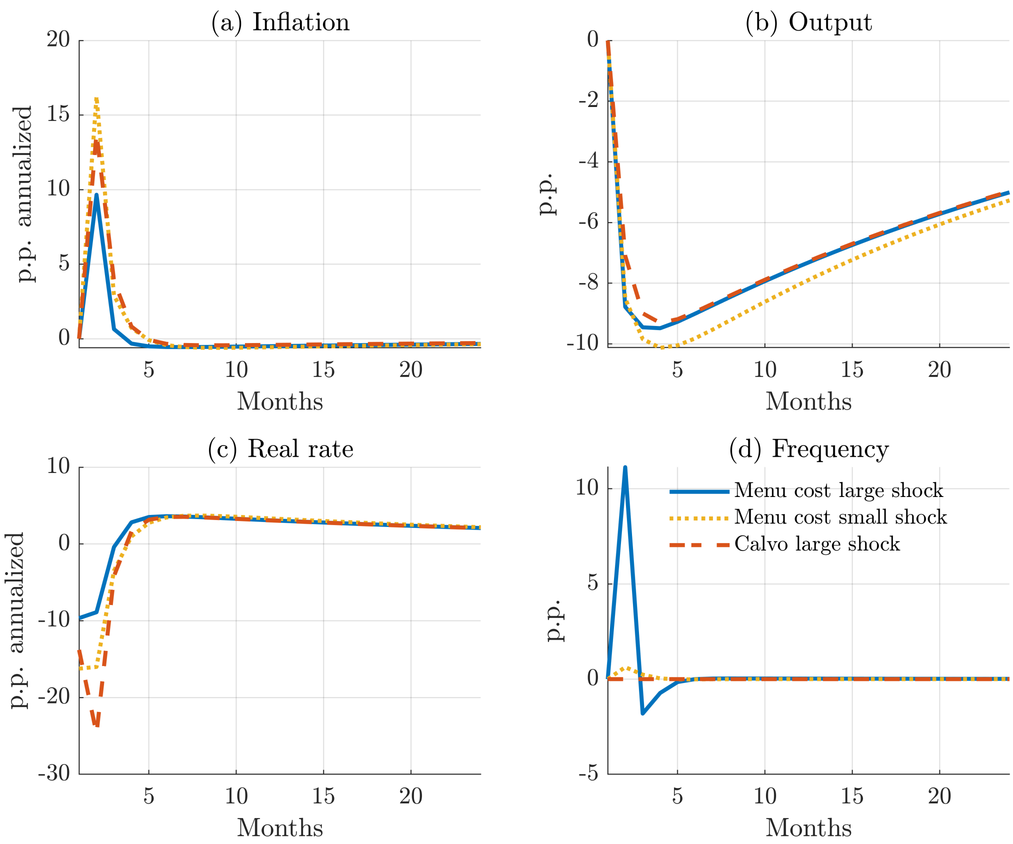

FIGURE 1. Impulse responses to a cost-push shock under the optimal monetary policy

The figure shows impulse responses in deviations from steady state to a large cost-push shock in the baseline menu cost model (blue solid line); and it contrasts the responses with those of a linearly-rescaled small cost push shock in the baseline model (yellow dotted line) and a large cost-push shock in the Calvo model (red dashed line).

Figure 2 displays the responses of key macro variables under optimal policy for a range of different adverse cost-push shock sizes in the menu cost model (blue solid line), in the Calvo model (red dashed line), and in a counterfactual menu-cost model (yellow dotted line) described below in Section 4.3. In particular, it draws the peak responses of inflation, output, and frequency, as well as the cumulative response of the annualized real interest rate over the first 2 years of the shock ($\sum_{t=1}^{24}(i_t - \pi_{t+1})/2$).

The peak frequency response in the baseline model (panel d) increases with the absolute shock size and has a zero slope around the steady state. This confirms the nonlinear nature of the optimal frequency response outlined above: frequency stays unresponsive to small shocks, but responds strongly to large shocks. The cumulative real rate figure (panel c) confirms that the policy is more aggressive in the menu cost model for large shocks than for small shocks and than in the Calvo model.[22] In line

[22]Though hard to perceive visually, the slope of the solid blue line in panel c is similar to that of the dashed red line when the shock size is close to zero. The cumulative real rate as a function of shock size is thus slightly convex in the menu cost model while concave in the Calvo model.

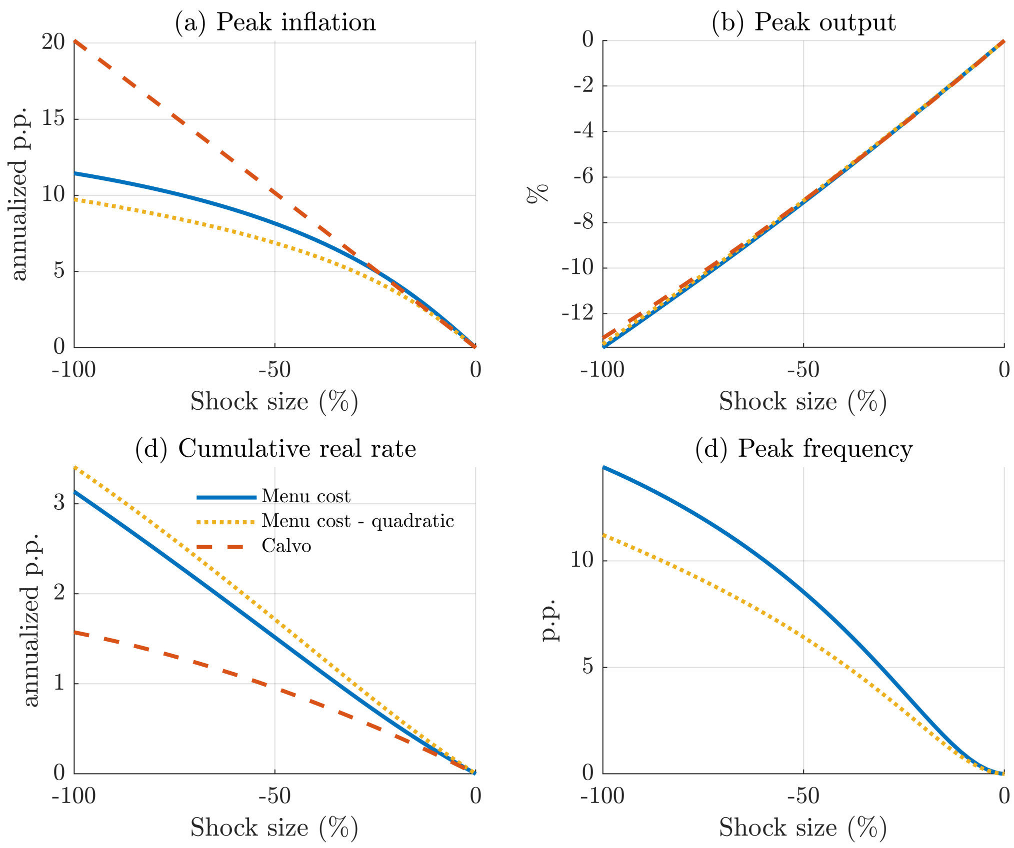

FIGURE 2. Optimal response to a cost-push shock for different shock magnitudes.

The figure displays the difference in the value of inflation, output gap, and repricing frequency between the period after the shock arrival and the value in the deterministic steady state. The real interest rate is evaluated over the first 2 years of the shock. For reference: a cost-push shock caused by the full removal of the subsidy (-100%) causes an increase in real marginal costs of around 5% on impact.

with the more aggressive policy in the menu cost model, the peak output effect is somewhat larger than in the Calvo model (panel b). Finally, the response of inflation increases less than proportionally with shock size (panel a) in the menu cost model, which is in contrast with the near-linearity of the inflation response of the Calvo model. When shocks are small, the optimal response of inflation in both models is near-linear and has a similar – though not identical – slope.

Figure 3 illustrates the relationship between inflation and the cumulative real rate under optimal policy for different cost-push shocks. The cumulative real interest rate can be interpreted as the "policy stance" to which the central bank commits in response to the inflationary cost push shock. When inflation is high, the central bank sets the nominal rate path so as to achieve an over-proportional increase in the cumulated real rate. In other words, when firms adjust prices more frequently, the policy stance responds more strongly to inflation.

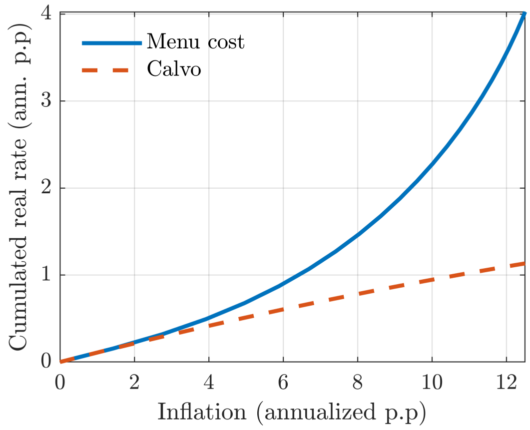

FIGURE 3. Strike while the iron is hot

The figure displays the relationship between impact inflation and the cumulative real interest rate under optimal policy and different cost push shocks. The real interest rate is evaluated over the first 2 years of the shock.

4.2. Inspecting the mechanism in the simplified model

In order to provide intuition for the driving forces behind the strong anti-inflationary stance of the central bank after large cost-push shocks, we analyze the simplified version of the model introduced in Section 2.5. As outlined there, the simplification introduces a subperiod of night, when the firms can reset their prices for free. This simplification transforms the dynamic problem into a series of static problems, but keeps the key channel active: firms decide endogenously when to adjust prices in response to the average shock and to monetary policy.

We cast the central bank's problem as a 2-dimensional optimization problem in output[23] and inflation, in analogy to the well-known textbook analysis of optimal monetary policy. A key advantage of the simplified model is that we can do this fully nonlinearly, without any need for approximation. We first inspect the central bank's choice set, defined by the Phillips curve, and then its objective.

Phillips curve. The central bank's choice set is given by the possible allocations consistent with a private equilibrium. We show first that these equilibria determine a relationship between inflation and output (or equivalently, the output gap): the Phillips curve. We characterize this relationship as a proposition and relegate its proof to the appendix. Throughout this section, we suppress the time subindex t for brevity.

PROPOSITION 1. *Private equilibria in the simplified model can be characterized by a single equation in inflation and output as follows*

$$1 = \int_{\mathsf{s}}^{\mathsf{S}} e^{(p)(1-\epsilon)}\frac{1}{\sigma}\phi\left(\frac{p+\pi}{\sigma}\right)dp + \left(\frac{\varepsilon\,(1-\tau)}{\varepsilon-1}Y\right)^{1-\epsilon}\left[1-\int_{\mathsf{s}}^{\mathsf{S}}\frac{1}{\sigma}\phi\left(\frac{p+\pi}{\sigma}\right)dp\right], \tag{30}$$

[23] The efficient output is $Y^e = 1$, so log output equals to log output gap.

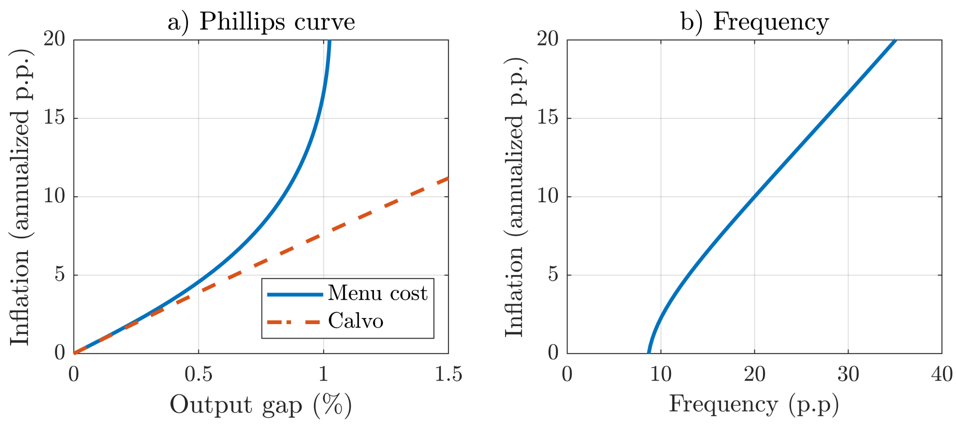

FIGURE 4. Phillips curve

Panel a plots the Phillips curve implicitly defined by equation (30), as well as the counterfactual value in the case of Calvo pricing. The output gap is $\log Y$ and annualized inflation is 12π. Panel b displays the mapping between frequency and inflation in the simplified model.

where s *and* S *are implicit functions.*[24]

PROOF. See Appendix A. □

Panel (a) of Figure 4 shows the Phillips curve in the calibrated model and compares it to the case of Calvo pricing. In both cases, the curves are increasing: under sticky prices, a policy easing that raises inflation would also raise output.[25] The key difference between Calvo and state-dependent pricing models is that while in Calvo the Phillips curve is near linear, the curve is convex in the menu cost model: that is, as inflation increases, its expansionary effect on output diminishes.

Panel (b) of Figure 4 depicts the relationship between the frequency of price changes and inflation in the menu cost model.[26] When inflation is low, the frequency of price changes remains close to its steady-state level, which is 8.7% per month in our calibration. Thus, locally the economy behaves similarly to a (suitably calibrated) Calvo economy (as emphasized by Auclert et al. 2024). However, as inflation gets larger, frequency rises as more and more firms decide to update their prices. This makes average prices more flexible, reducing the responsiveness of output to changes in inflation, thus steepening the Phillips curve.[27]

[24] $\mathsf{S}(Y,\tau)$ and $\mathsf{s}(Y,\tau)$ are the two roots of the equation $\left(\frac{\epsilon}{\epsilon-1}(1-\tau)Y\right)^{1-\epsilon} - ((1-\tau)Y)^{1-\epsilon} - \eta = \mathsf{x}(Y,\tau)^{1-\epsilon} - (1-\tau)Y\mathsf{x}(Y,\tau)^{-\epsilon}$ for $\mathsf{x} = \mathsf{s}, \mathsf{S}$.

[25] At zero inflation, the slopes are identical in the two models. This happens by construction, as the Calvo model is re-calibrated to replicate the slope of the Phillips curve in the limit as inflation approaches its steady state value.

[26] Frequency is exogenous and independent of inflation in the Calvo model (not shown).

[27] At some point, in our calibration at frequencies over 40% per month, and roughly corresponding to annualized inflation levels around 20%, the Phillips curve becomes backward bending (not shown). At this level, monetary policy reach its maximum effectiveness in stimulating activity, and any further inflation reduces output. However, such levels of inflation and frequency are fairly extreme. For the rest of the analysis we restrict our attention to inflation levels that can be large, but not as large as to go beyond this point.

The Phillips curve describes the choice set of the policymaker setting inflation. Its slope reflects the state-dependence of the inflation-output trade-off involved in monetary policy decisions: It states how much the output gap must decline to reduce inflation by a percentage point, also known as the *sacrifice ratio* of monetary policy. This slope more than doubles when the frequency reaches 20% per month, a magnitude documented during the post-COVID inflation surge (Montag and Villar 2023). While in a low-frequency and low-inflation environment the sacrifice ratio is high, it becomes much lower once frequency and inflation increase.[28]

Welfare. We turn next to the central bank's preferences in the simplified model. The central bank maximizes welfare (1), which in the simplified model is equivalent to maximizing period utility $U = \log C - N$.[29] First, we describe how the underlying welfare distortions, namely misallocation and price-adjustment costs, affect utility. Second, we link these welfare distortions to the output gap and inflation in the simplified model.

PROPOSITION 2. *Let $U - U^e$ be the central bank's utility gap relative to the utility under efficient allocation expressed in efficient-consumption-equivalent units. Let the welfare-relevant markup be the relative price of firm j divided by the welfare-relevant marginal cost: $\mu(j) = \frac{P(j)/P}{WRMC(j)}$, where $WRMC(j) \equiv wA(j)/A$. Then the utility gap can be expressed as a function of the average welfare-relevant markup ($\overline{\mu}$), the markup dispersion (ζ^{μ}), and price adjustment costs as*

$$U - U^e = \underbrace{\underbrace{-\log\overline{\mu} - \left(\frac{1}{\overline{\mu}} - 1\right)}_{\text{Average markup}} - \frac{1}{\overline{\mu}}\underbrace{(\zeta^{\mu} - 1)}_{\text{Markup dispersion}}}_{\text{Misallocation}} - \underbrace{\eta g^0}_{\text{Adjustment costs}}, \tag{31}$$

where the average welfare-relevant markup is $\overline{\mu} \equiv \left(\int \mu(j)^{1-\epsilon} dj\right)^{\frac{1}{1-\epsilon}}$, the markup dispersion is $\zeta^{\mu} \equiv \int (\mu(j)/\overline{\mu})^{-\epsilon} dj$, and ηg^0 are the price adjustment costs in labor units.

PROOF. See Appendix B. □

Proposition 2 shows that welfare costs are intuitively driven by two components: First, the *misallocation*, which is caused by the deviation of firms' relative prices from the welfare-relevant marginal costs, introduces a labor wedge. Misallocation can be further decomposed into the "average markup" term which is a nonlinear function of the average welfare-relevant markup ($\overline{\mu}$), and the product of the inverse average markup and the "markup dispersion". The average welfare-relevant markup describes *average* over- or under-consumption, while the markup dispersion refers to the inefficient *relative* consumption of different good varieties. Second, labor is inefficiently

[28]Blanco et al. (2024b) also discuss how the sacrifice ratio changes with the level of inflation.

[29]This is because private equilibrium conditions are static in the simplified model. Note that for the same reason there is no difference between commitment and discretion.

allocated to price adjustment (*menu costs*), which is captured by the third component.[30]

In the simplified model, the utility gap and its components can be expressed as functions of inflation and output (gap). This is analogous to the Calvo case, where welfare can also be expressed as a function of the output gap and inflation.[31]

This is formulated in Proposition 3.

PROPOSITION 3. *In the simplified model, utility gap can be expressed as*

$$
\begin{aligned}
U - U^e \quad = \quad & \underbrace{\log(Y) - (Y-1)}_{\textit{Average markup}} - \\
& \underbrace{Y\left(\int_{\mathrm{s}}^{\mathrm{S}} e^{p(-\epsilon)} \frac{1}{\sigma}\phi\left(\frac{p+\pi}{\sigma}\right) dp + \left(1 - \int_{\mathrm{s}}^{\mathrm{S}} \frac{1}{\sigma}\phi\left(\frac{p+\pi}{\sigma}\right) dp\right) e^{\mathrm{p}^*(-\epsilon)} - 1\right)}_{\textit{Markup dispersion}} - \\
& \underbrace{\eta\left[1 - \int_{\mathrm{s}}^{\mathrm{S}} \frac{1}{\sigma}\phi\left(\frac{p+\pi}{\sigma}\right) dp\right]}_{\textit{Adjustment costs}}
\end{aligned}
\tag{32}
$$

where s, S *and* p^* *are implicit functions of inflation.*[32] *The utility function depends only on inflation and output: the first term driven by the average welfare-relevant markup in equation* (32) *depends only on the output (gap), whereas price dispersion and adjustment costs depend only on inflation.*

PROOF. See Appendix A. □

Figure 5 illustrates this decomposition for the simplified menu cost model and contrasts it to the analogous decomposition in Calvo. The average welfare-relevant markup coincides in the two models (panel b). Price dispersion increases in inflation in Calvo, while it mildly decreases in inflation in the menu cost model (panel c). This decrease is due to the endogenous increase in the frequency of price changes, which leads more firms to close their markup gaps. Furthermore, it is exactly the firms with the largest markup gaps, who endogenously self-select to adjust. In contrast, the adjustment costs (panel d) increase with inflation in the menu cost model in line with the endogenous increase in the frequency. Adding up the latter two pricing frictions and comparing them to price dispersion in Calvo, we see that the welfare effects of nominal rigidities are U shaped in inflation in both models. Quantitatively, however, the losses from nominal rigidities are somewhat smaller in the menu cost model (Burstein and Hellwig 2008). Thus, the central bank is slightly less inflation-averse than in the case of Calvo pricing, which is reflected in the different degrees of ellipticity of the iso-welfare curves shown in panel a.

[30] The welfare decomposition into distortions of Proposition 2 straightforwardly generalizes to the full model. It also applies to Calvo model, in which case the last term is 0.

[31] In particular, in Calvo, up to a second order the utility function is quadratic of the form $-\frac{1}{2}\left[\hat{y}^2 + \epsilon\left(\frac{1-\theta}{\theta}\right)\hat{\pi}^2\right]$, where the 'hat' denotes deviation from the zero inflation steady state (Woodford 2003).

[32] $\mathrm{s}(\pi)$, $\mathrm{S}(\pi)$ and $\mathrm{p}^*(\pi)$ solve the Ss band conditions, and the definition of the price level.

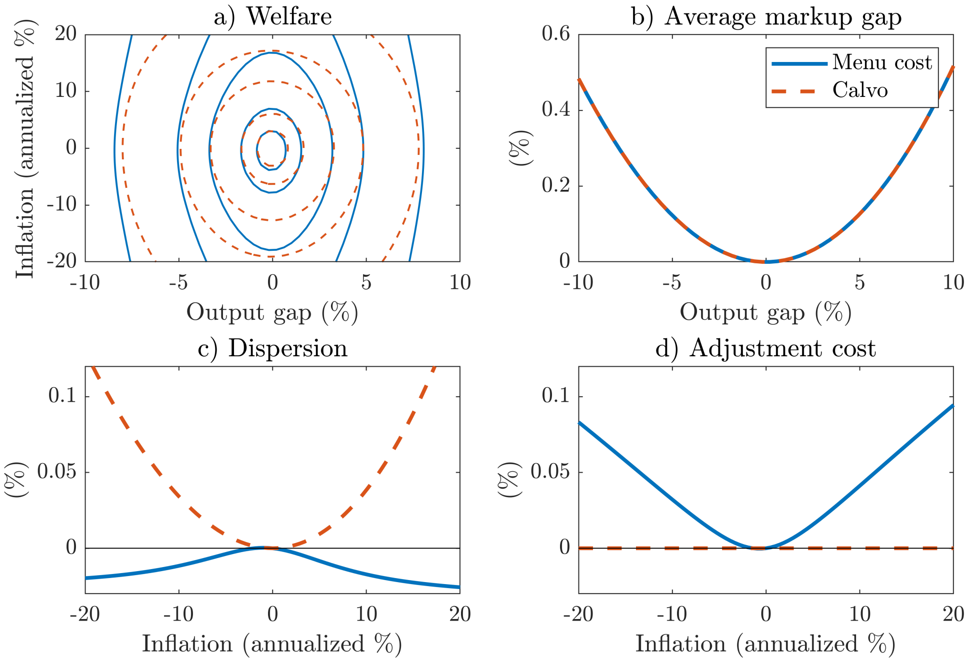

FIGURE 5. Welfare decomposition

Note: Decomposition of welfare differences according to equation (31). Welfare gaps are expressed in % of efficient consumption.

Optimal policy. We can now set up the central bank's problem. It chooses inflation π and output Y so as to maximize the objective (32) subject to the Phillips curve (30). Figure 6 represents this problem and its solution graphically. It shows the Phillips curve (PC, dashed lines) for a particular value of the exogenous cost-push shock τ; and the utility isoquant (thin solid lines) that is tangent to that PC. The optimal policy is defined by their tangential point, points A and B for Calvo and menu cost pricing, respectively. The "target rule" traces these points for different levels of the cost-push shock (thick solid line).

Three key insights can be derived from this figure. First, the slope of the target rule at zero is slightly smaller in the menu cost economy than in the Calvo economy. Since the Phillips curve slopes coincide for both models at zero by construction, the different slope of the target rule is exclusively due to the fact that the welfare function is less anti-inflationary in the menu cost model. This effect is quantitatively small.

Second, the target rule is almost linear under Calvo pricing (red). This is the consequence of an almost linear Phillips curve and a welfare function that is approximately quadratic. Under menu cost pricing, however, the target rule is concave (blue). This implies that the central bank leans more and more aggressively against inflation as inflation increases. The central bank *strikes while the iron is hot.*

This begs the question of why. Calvo and menu cost models differ in both the objective and the constraint. Which one is responsible for this? To shed light on this question, we compute optimal policy assuming that the central bank faces the Phillips

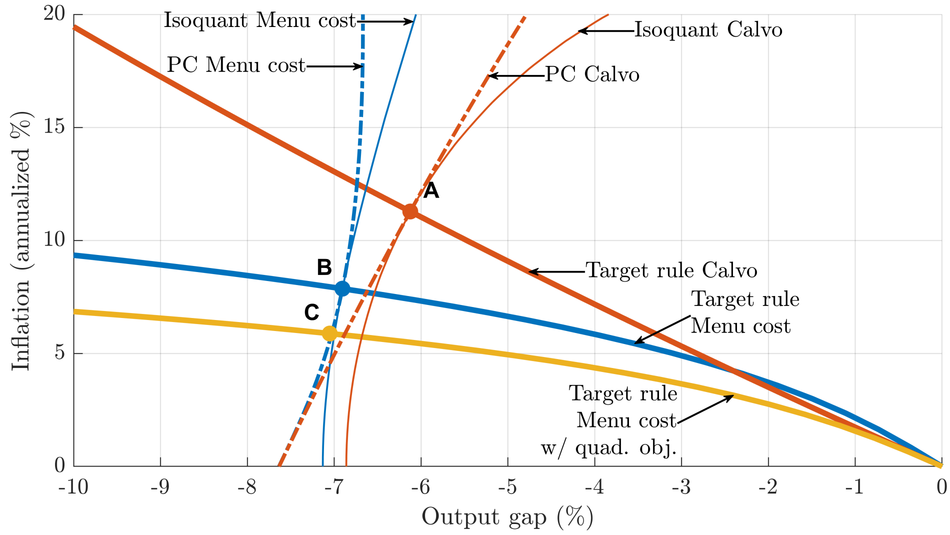

FIGURE 6. Optimal policy and the target rule

This figure combines the welfare functions from panel a in Figure 5 with the Phillips curves from panel a in Figure 4 to derive the target rule.

curve of the menu cost framework, but it has the objective over inflation and output of the Calvo model. That is, we look for tangential points of the blue Phillips curve with the red iso-welfare curve. Point C marks this point for the given Phillips curve. The yellow line traces these points out for all levels of the cost-push shock. Since the Calvo central bank is slightly more inflation averse, the yellow line is lower than the blue menu-cost target rule. Yet, the degree of concavity is similar. We can thus conclude that the nonlinearity of the target rule in the menu cost model is driven by the strong convexity of the Phillips curve.[33] The shape of the objective function, if anything, diminishes the concavity of the target rule a bit, relative to Calvo. This is the third insight.

What is the intuition behind the strike while the iron is hot result? In the case of small shocks, the change in the frequency of price adjustments is negligible, and thus the logic of the Calvo framework still applies: The central bank tolerates some inflation to partially cushion the fall in the output gap. However, as inflation rises, frequency starts to pick up and prices become more flexible. This reduces the sacrifice ratio: to achieve the same impact on the output gap, the central bank would need to let inflation increase substantially more in this case, and it is not willing to do so. Thus, after a large cost-push shock, the central bank stabilizes inflation more relative to the output gap than after small shocks. The central bank "leans against frequency," tightening policy more aggressively in the case of a large shock that increases frequency. In the nonlinear Calvo model (red dashed line), by contrast, the nonlinearity is negligible, despite the fact that we do not linearize the model.

[33]The cost-push shock itself causes a parallel sideward shift of the Phillips curve (Proof: see appendix A). It is thus not a cause for the nonlinearity of the target rule.

Note that the strike while it's hot result is not obvious. The shape of the objective function could have fully offset the nonlinearity of the Phillips curve such that the target rule remained linear. Indeed, in the linearized Calvo framework, for example, the change in the objective fully offsets the change in the shape of the Phillips curve caused by *permanent* changes in the frequency of price changes. There, even though the slope of the Phillips curve increases with the frequency, the relative weight of inflation in the objective declines proportionally (Galí 2008). It declines because higher frequency raises the flexibility of the price level, and reduces the increase in price dispersion caused by a marginal increase in inflation. The two effects fully offset each other and the slope of the target rule depends only on the elasticity of substitution $(-1/\varepsilon)$ and not on frequency. We conclude that the relative stability of the objective function in the presence of large shocks is just as important for our strike it while it's hot result as the nonlinearity of the Phillips curve.

4.3. Relation to the full model

In the nonlinear full dynamic model, the Phillips "relationship" is a dynamic multidimensional relationship, depending both on current and expected state variables and is described by several equations. There is thus no simple structural relationship linking current inflation and output any more.[34] Instead, the Phillips relationship is made up of a dynamic block of equations which contains not only the definition of the price level (22), the firms' optimality conditions (now dynamic) (14-16) and the definition of frequency (21) as in the static model, but also the law of motion of the distribution (20), and the value function (17). A similar argument applies to the welfare function in terms of current inflation and output. The table in the Appendix E compares the equilibrium conditions of the simplified (static) Calvo and menu cost models to those of the full model.

Nevertheless, much of the intuition carries over and it is still useful to think broadly in terms of objective and constraints. To illustrate this, Figure 7 compares the Phillips curve and the target rule from the simplified model with the analogous relationships between output and inflation implied by the full model after a cost push shock under optimal policy (panel a) and after a monetary policy shock under a Taylor rule (panel b). In all cases we display the response of variables to shocks of different magnitudes on impact starting from the steady state of the Ramsey problem. As explained above, in the full model these relationships are not structural, but are conditional on the initial conditions and the shock process. Nevertheless, these two relationships are fairly stable with respect to those conditions and are surprisingly similar to the structural relationships uncovered from the simplified model.

Interpreting this figure for the full model, two features are worth noticing. First, the slope of the output-inflation relationship under optimal policy (what we call "the

[34]Note that the same is true in the nonlinear Calvo model, the Phillips curve only emerges under linear approximation.

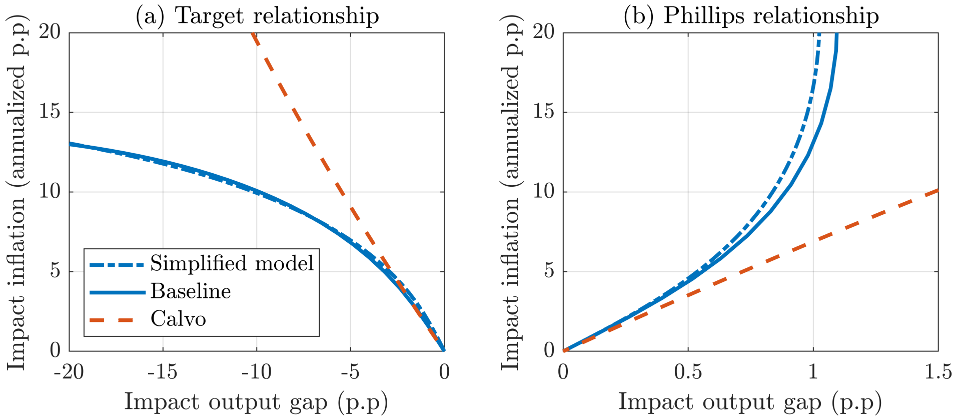

FIGURE 7. Simplified versus full model

Note: Panel a contrasts the target rule in the simplified model (blue dash-dotted line) with the response of inflation and output gap on impact in the full menu cost (blue solid line) and Calvo models (red dashed line) to a cost-push shock under optimal monetary policy. Panel b contrasts the Phillips curve in the simplified model with the impact response to a monetary policy shock under a Taylor rule in the full menu cost and Calvo models.

target rule") is almost indistinguishable at zero from that under Calvo (panel a), which, is given by $-1/\epsilon$ up to a first order (see Galí 2008). The Calvo model thus delivers a good approximation of optimal policy for moderate levels of inflation.

Second, the nonlinearity of the menu cost model becomes quantitatively significant quite quickly. At 10% inflation, for example, the slope of the Phillips relationship is 150% larger than under Calvo pricing. At the same inflation level, the optimal policy response to a cost push shock is almost 50% more restrictive in terms of output. Thus, while at moderate inflation levels the Calvo model is a good enough approximation of the menu cost model, this equivalence breaks down at inflation levels such as those seen in 2022, when inflation reached approximately 10 percent.

The nonlinearity of the sacrifice ratio is the main reason behind the "strike while the iron is hot" result also in the full model. To show this, we have rerun our optimal Ramsey policy exercise in the full model combining the menu cost framework with a counterfactual quadratic objective in the (i) inflation gap, the deviation of inflation from its optimal steady state value, and (ii) output gap with relative weights derived from the second-order approximation of the Calvo model (Woodford 2003). The objective approximates the true objective in the nonlinear Calvo framework well. The results are shown in Figure 2 (yellow dotted line). The figure shows that, in line with the results of the analogous exercise in the simplified model, the inflation response is similar, even more nonlinear under this counterfactual scenario than the baseline. This confirms that the key reason behind the nonlinearity of the target relationship is the nonlinearity of the Phillips relationship also in the full model.

4.4. Robustness and sensitivity analysis

We now show the robustness of the nonlinear "strike while the iron is hot" optimal monetary policy. In particular, we explore its robustness in an extension of the Golosov and Lucas (2007) model, the CalvoPlus model (Nakamura and Steinsson 2008), and its robustness to alternative parameter choices.

CalvoPlus model. The CalvoPlus model is a variation of the canonical menu cost model, where the menu cost is stochastic: price adjustment is free with probability α, as in the Calvo (1983) model, and takes a positive value (η) with probability $1 - \alpha$. This setup introduces small price changes, and therefore, improves the model's ability to fit better the distribution of price changes. At the macro level, by reducing the selection of large price changes, it increases the real effects of monetary policy and brings it closer to time-series evidence (Nakamura and Steinsson 2010). In the context of our analysis, the framework affects the nonlinearity of the Phillips relationship, which raises concerns that it may potentially modify the monetary policy prescriptions already discussed. However, we show that this is not the case. This result supports the robustness of the strike-while-the-iron-is-hot policy conclusion in a realistic extension of the canonical Golosov-Lucas model.

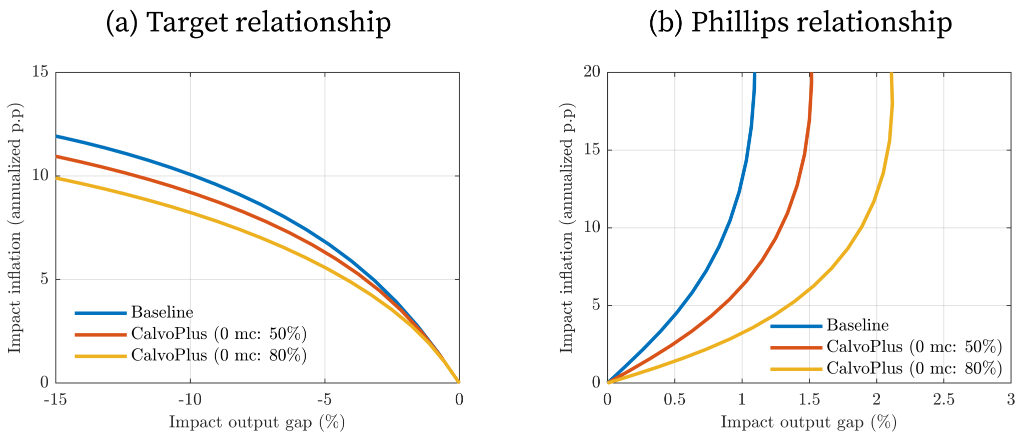

FIGURE 8. Target and Phillips relationships in the baseline and in the CalvoPlus model

The figures contrast the relationship between inflation and output gap movements on impact in the baseline model versus the CalvoPlus model under optimal policy and various sizes of cost-push shocks (target relationship, panel a) and under a Taylor rule and various sizes of monetary policy easing shocks (Phillips relationship, panel b). Two parameterizations of the CalvoPlus model are presented: (i) the probability of a zero menu cost is 50%; and (ii) 80%.

Two parameterizations are presented: (i) the probability of zero menu cost α is 50%; and (ii) 80%. We recalibrate the menu cost η and the dispersion of idiosyncratic shocks σ parameters to match in the steady state the average frequency in the U.S. data of 8.7%, and match a 20% frequency increase at a 10% inflation rate.

Results are presented in Figure 8. Panel (a) of the figure shows the target relationship between the inflation rate and the output gap on impact after cost push shocks under optimal policy. The figure shows that the relationship is robust in the CalvoPlus model: strike-while-the-iron-is-hot policy is still optimal. Notably, the extent of nonlinearity *increases* with the Calvo parameter (α): the higher the probability of free price changes, the more anti-inflationary optimal policy should be. This happens, even though, as we have seen above, the target relationship is near linear in the Calvo model. A key factor in this result is that the Calvo parameter (α) in the CalvoPlus model brings the baseline menu cost model closer to the Calvo model for small shocks, but influences its response to large shocks much less, when the frequency-response is driven by the behavior of the $(1 - \alpha)$ firms facing positive menu costs.

Panel (b) shows the relationship between inflation and output gap under a Taylor rule on impact after a monetary policy shock in the three models - the Phillips relationship discussed above. The figure shows that the CalvoPlus model can substantially reduce the slope of the Phillips relationship, bringing the model closer to the Calvo (1983) model after small shocks, also improving the realism of the framework (Nakamura and Steinsson 2010). At the same time, the CalvoPlus framework still implies a highly nonlinear Phillips relationship, with a slope that increases *even faster* than the baseline for similar increases in the inflation rate. This happens because, as the shocks become larger and the frequency of price adjustment increases, the share of price-adjusters paying the adjustment costs increases mechanically, bringing the model closer to the canonical Golosov and Lucas (2007) framework. The increase in this share leads to an additional source of nonlinearity in the Phillips relationship. This higher nonlinearity further reduces the sacrifice ratio of disinflation in CalvoPlus models with higher Calvo parameters, thus making a stricter anti-inflationary stance optimal.

Alternative parameterizations. Panel (a) of Figure 9 contrasts the target relationship between annualized inflation and the output gap (on impact) under Ramsey optimal policy in the baseline model with alternatives with varying degrees of persistence of the cost-push shock ($\rho_\tau = 0.75, 0.99$); and with alternative values of the elasticity of substitution parameter ($\epsilon = 3, 11$). It also shows straight lines with slope $-1/\epsilon$ for $\epsilon = 3, 7, 11$. The figure shows that (i) the target relationship is influenced by the persistence of the underlying shock, but the variation is quantitatively small. Furthermore, (ii) the elasticity of substitution plays a key role in determining the slope of the target relationship. For small shocks, the slope of the target relationship is quantitatively close to, albeit slightly higher than $-1/\varepsilon$, which is the slope of the target rule, the relationship between inflation and the change in the output gap, in the linearized Calvo model. Lastly, (iii) the qualitative features of the nonlinearity after large shocks are robust: it is optimal to strike while the iron is hot for a wide range of parameter values.

Panel (b) of Figure 9 shows the robustness of the Phillips relationship. The figure reports the relationship between the impact effect of annualized inflation and the

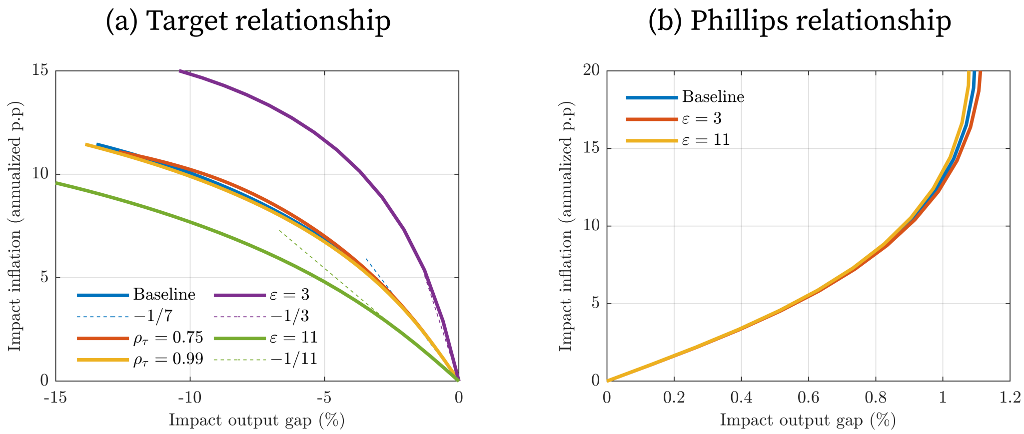

FIGURE 9. Robustness of the target and Phillips relationships for alternative parameters

The figures recreate the relationship between inflation and output gap under optimal policy on impact after cost-push shocks (target relationship, panel a) and Taylor rule on impact after monetary policy shocks (Phillips relationship, panel b) for alternative parameter values. They show sensitivity to various elasticity of substitution parameters ($\epsilon = 3, 11$) and various persistence values of the cost push shock ($\rho_\tau = 0.75, 0.99$).

impact output gap for different i.i.d. monetary policy shocks of varying sizes under the Taylor rule as panel (b) of Figure 7. It reports how the relationship changes when varying the elasticity of substitution parameter $\epsilon = 3, 11$.[35] The figure shows that the relationship is robust and stays nonlinear across the relevant parameter space.

5. Optimal monetary policy: additional results

We now proceed to investigate additional results: optimal long-run inflation, optimal monetary policy to an aggregate productivity shock, and time-inconsistency of the Ramsey optimal monetary policy.

5.1. The steady state under the optimal policy

The solution of the Ramsey planner's problem has a steady state featuring a slightly positive inflation of 0.07%.[36] This is different from the standard New Keynesian model with Calvo pricing (Galí 2008), where the optimal inflation in the Ramsey steady state is zero. The value of inflation in the Ramsey steady state in the menu cost model is very close to the value of steady-state inflation that maximizes steady-state welfare, which in turn is also very close to the value of inflation that minimizes the frequency of price adjustments.

What explains the positive optimal inflation? The key factor is the asymmetry of the profit function (18). For a firm, a negative price gap is more undesirable than a positive price gap of the same size because a negative price gap $-x$ leads to a much

[35] We recalibrate the menu cost and the idiosyncratic quality shock volatility such that the steady state frequency stays constant across calibrations and it generates 20% frequency at 10% inflation.

[36] In our numerical exploration, we have only found a single steady state.

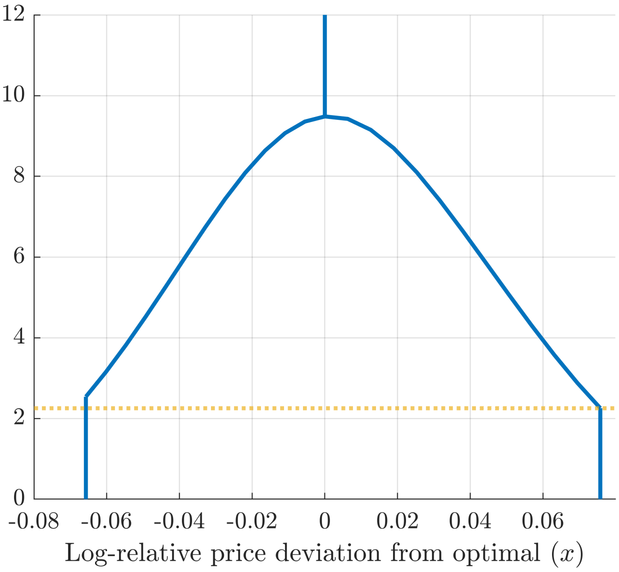

FIGURE 10. Steady-state price-gap density.

The figure displays the steady-state price-gap density $g(x)$ with zero inflation. The dashed yellow line indicates the mass of firms at the upper threshold of the (S, s) band.

larger sales increase at a markup loss of $-x$, while the positive price gap x leads to only somewhat smaller sales drop at a markup gain of x. This implies that the (S, s) band is asymmetric: the lower threshold s_t is closer to the optimal price than the upper one S_t (see Figure 10). Thus, in the zero inflation steady state, there is more mass of firms close to the lower threshold of the inaction band than to the upper threshold. As a result, there are more upward than downward price adjustments. Small positive inflation raises the optimal reset price p^* and shifts the (S, s) band leftwards and thus reduces the number of upward price movements by more than it increases the number of downward price movements. The frequency of price adjustments decreases and, with it, the distortions caused by menu costs. Quantitatively, this effect is small but not negligible.

5.2. Timeless optimal monetary response to TFP shocks

Next, we consider TFP shocks, which affect the efficient allocation. In the standard New Keynesian model with Calvo prices, the response to such shocks is characterized by strict price stability: the central bank steers real interest rates to replicate the path of natural interest rates, which leads to inflation and the output gap remaining at zero. This is commonly known as the "*divine coincidence*" (Blanchard and Galí 2007).

A version of the divine coincidence also holds in our economy.[37] As we have shown in Section 5.1, the Ramsey plan features a positive level of trend inflation in the long run. In response to a TFP shock, optimal policy keeps inflation constant at this level:

[37] We thus generalize to the case of heterogeneous firms the finding of Nakov and Thomas (2014) of a divine coincidence in response to TFP disturbances when pricing is state-dependent.

PROPOSITION 4. *The timeless Ramsey policy keeps inflation constant at steady state level in response to aggregate TFP shocks.*

PROOF. See Appendix C. □

As inflation remains constant, the frequency of repricing and the price gap distribution also stay constant. Strict targeting of the optimal steady-state inflation rate thus simultaneously minimizes inefficient output fluctuations (the average markup gap) and the costs of nominal rigidities (markup dispersion and adjustment costs). Notice that the shape of the Phillips curve plays no role in this result, and thus the prescription is the same for small and large shocks.

5.3. Time-0 problem

We now turn to investigating the time inconsistency of optimal policy. To assess its magnitude, we solve the optimal policy problem, starting from the price distribution in the Ramsey steady state, assuming that the central bank faces no previous pre-commitment. In this case, the Lagrange multipliers associated with forward-looking equations are initially set to zero. This problem is often referred to as the "time-0 problem" (Woodford 2003).

The solid blue lines in Figure 11 show the time path under the optimal policy. The labor subsidy is set to zero in this exercise, which, therefore, ceases to offset any markup distortions caused by the firms' market power. The steady state of the Ramsey policy is time-inconsistent: without pre-commitment, the central bank engineers a temporary expansion. Thereby, it raises welfare by bringing output closer to its efficient level at a cost of elevated pricing distortions arising from the higher inflation.

The dashed red line on Figure 11 shows the equivalent time-0 response in the Calvo model. The figure shows that the incentive to surprise is substantially weaker in the menu cost model: both the inflation and output gap increases are smaller relative to the Calvo model. The reason is that the price level becomes more flexible in the state-dependent model: the unexpected easing causes a sizable inflation spike, which causes an increase in the frequency of price changes. As a result, the output gap increases by less than it would under exogenous frequency. That is, the output boost from a given amount of inflation is lower than under Calvo. Since, as we saw before, the central bank's objective function isn't significantly different than under Calvo, the central bank thus eases less aggressively.

There is a countervailing force that raises the time inconsistency in our baseline model relative to the Calvo model. Namely, due to the idiosyncratic shocks, the uniform labor subsidy of $\tau = 1/\epsilon$ is insufficient to fully offset the markups for all firms in the steady state, as it does in the Calvo model. A time-0 optimal policy, therefore, stays time inconsistent even with a $\tau = 1/\epsilon$ labor subsidy (not shown). The optimal policy easing in this scenario, however, is two orders of magnitude smaller than those under

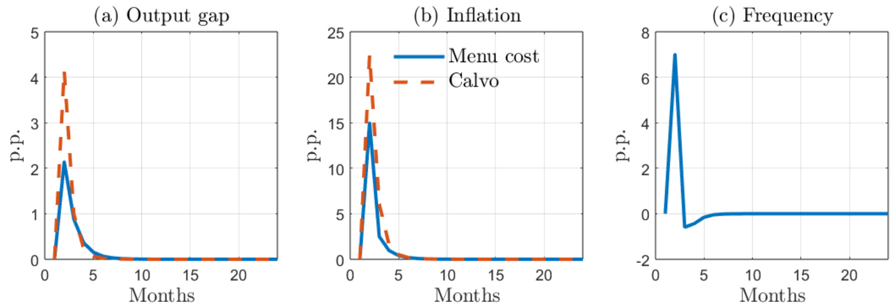

FIGURE 11. Time inconsistency of the optimal policy.

The figure compares the time-0 optimal policies in the menu cost model and in the Calvo model. Inflation is annualized 12π.

no labor subsidy. Therefore, this channel is too weak to counteract the opposite effect caused by the more flexible price level detailed above.

A corollary to the negligibility of the time inconsistency with an appropriate labor subsidy is that the analysis in the previous sections, where we adopted a timeless perspective, would go through without any quantitatively relevant changes also if we adopted a time-0 perspective.

6. Conclusion

This paper characterizes the Ramsey optimal monetary policy in a canonical menu cost model. We find that in the presence of large cost-push shocks, optimal monetary policy should commit to mitigating inflation more aggressively than what the standard New Keynesian model prescribes. The central bank exploits the endogenous reduction in the sacrifice ratio brought about by the increase in price flexibility in order to contain inflation more. That is, it *strikes while the iron is hot*. Importantly, this nonlinearity can be quantitatively relevant already at moderately elevated inflation rates such as those seen during the recent inflation surge. This policy prescription diverges markedly from that of the standard New Keynesian model with exogenous timing of price adjustment, which fails to capture such nonlinear dynamics. When confronted with TFP shocks, our findings indicate that the optimal policy in the menu cost model involves a commitment to full price stability, akin to the standard New Keynesian model.

In sum, our research underscores the importance of an aggressive anti-inflationary policy by the central bank in the face of large shocks. By committing to policies that curb inflation and stabilize the repricing frequency, the central bank can deliver a more favorable macroeconomic outcome. Our analysis is confined to the case of nominal price rigidities in the canonical menu cost models of Golosov and Lucas (2007) and Nakamura and Steinsson (2010); we leave for future research the interaction with wage rigidities and assessment of optimal policy in more complex and realistic price-setting frameworks.

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APPENDIX

Appendix A. The simplified model

This appendix lays out the simplified model and provides the proofs for the related propositions.

Model description. As explained in the main text in Section 2.5, prices are reset overnight in the simplified model. Thus, all dynamics are muted, such that the model boils down to a sequence of static models.[38] For this reason, we remove the time subscript in the simplified model.

As explained there, and after removing a number of trivially redundant equations and variables, the model boils down to the following eight equations, which define an equilibrium in nine variables $w, \pi, C, N, s, S, g^0, g^c(x), p^*$, leaving the policymaker with one degree of freedom to choose π:

$$
\begin{aligned}
e^{p^*} &= \frac{\epsilon}{(\epsilon-1)}(1-\tau)w, && \text{(A1)}\\
(e^{p^*})^{1-\epsilon} &- (1-\tau)w(e^{p^*})^{-\epsilon} - \eta = (e^{(p^*+s)})^{1-\varepsilon} - (1-\tau)w(e^{(p^*+s)})^{-\epsilon}, && \text{(A2)}\\
(e^{p^*})^{1-\epsilon} &- (1-\tau)w(e^{p^*})^{-\epsilon} - \eta = (e^{(p^*+S)})^{1-\varepsilon} - (1-\tau)w(e^{(p^*+S)})^{-\epsilon}, && \text{(A3)}\\
g^c(x) &= \frac{1}{\sigma}\phi\left(\frac{x+\pi+p^*}{\sigma}\right), && \text{(A4)}\\
w &= C, && \text{(A5)}\\
g^0 &= 1 - \int_s^S g^c(x)dx, && \text{(A6)}\\
1 &= \int_s^S e^{(x+p^*)(1-\epsilon)} g^c(x)\, dx + g^0 e^{p^*(1-\epsilon)}, && \text{(A7)}\\
N &= C\left(\int_s^S e^{(x+p^*)(-\epsilon)} g^c(x)\, dx + g^0 e^{p^*(-\epsilon)}\right) + \eta g^0. && \text{(A8)}
\end{aligned}
$$

In the above, as in the main text, we define the distribution and value functions as a function of the price gap x, as is common in the state-dependent pricing literature. However, for the analysis of the simplified model, it is convenient to rather define them as a function of the price level $p \equiv x + p^*$. The Ss bands will also be re-normalized accordingly. After this change of variable, the system reads:

[38] Alternatively, the static model version can be seen as a particular case of the complete model in which we set $\beta = 0$ and assume that the initial distribution is such that all firms have set the same price last period ($g^c_{-1}(x) = 0$, $g^0_{-1} = 1$, $p^*_{-1} = 1$)

$$e^{p^*} = \frac{\epsilon}{(\epsilon - 1)}(1-\tau)w, \tag{A9}$$

$$(e^{p^*})^{1-\epsilon} - (1-\tau)w(e^{p^*})^{-\epsilon} - \eta = (e^{s})^{1-\varepsilon} - (1-\tau)w(e^{s})^{-\epsilon}, \tag{A10}$$

$$(e^{p^*})^{1-\epsilon} - (1-\tau)w(e^{p^*})^{-\epsilon} - \eta = (e^{S})^{1-\varepsilon} - (1-\tau)w(e^{S})^{-\epsilon}, \tag{A11}$$

$$g^{c}(p) = \frac{1}{\sigma}\phi\left(\frac{p+\pi}{\sigma}\right), \tag{A12}$$

$$w = C, \tag{A13}$$

$$g^{0} = 1 - \int_{s}^{S} g^{c}(p)dp, \tag{A14}$$

$$1 = \int_{s}^{S} e^{p(1-\epsilon)} g^{c}(p)\,dp + g^{0} e^{p^*(1-\epsilon)}, \tag{A15}$$

$$N = C\left(\int_{s}^{S} e^{p(-\epsilon)} g^{c}(p)\,dp + g^{0} e^{p^*(-\epsilon)}\right) + \eta g^{0}. \tag{A16}$$

Note that the firm's decisions are much simpler than in the full model, since they are static. The reset price maximizes firms' current profits by setting a constant markup (A9). Furthermore, firms keep their (logged quality-adjusted real) price p unchanged as long as current profits exceed profits under the optimal price p^* minus the menu cost, that is when $\Delta\Pi(p) \equiv \Pi(p) - (\Pi(p^*) - \eta w) > 0$. $\Delta\Pi(p)$ has exactly 2 positive roots.[39] One root is smaller than $p^*(s)$, and one root is larger than $p^*(S)$ and the function $\Delta\Pi(x)$ is positive between them. Thus, these two roots define the Ss bands, which characterize the optimal update decision.

Phillips curve. We now show how we derive the Phillips curve displayed in Proposition 1. First, we use equations (A12)-(A14) to eliminate g^0, w and $g^c(p)$ from equations (A9), (A10), (A11) and (A15). Then we use the resulting version of (A9) to eliminate p^* in the remaining 3 equations. This leaves us with the following equations:

$$\left(\frac{\epsilon}{\epsilon-1}(1-\tau)C\right)^{1-\epsilon} - ((1-\tau)C)^{1-\epsilon} - \eta = e^{S(1-\epsilon)} - (1-\tau)Ce^{S(-\epsilon)}, \tag{A17}$$

$$\left(\frac{\epsilon}{\epsilon-1}(1-\tau)C\right)^{1-\epsilon} - ((1-\tau)C)^{1-\epsilon} - \eta = e^{s(1-\epsilon)} - (1-\tau)Ce^{s(-\epsilon)}. \tag{A18}$$

$$1 = \left[\int_{s}^{S} e^{(p)(1-\epsilon)} \frac{1}{\sigma}\phi\left(\frac{p+\pi}{\sigma}\right)dp + \left(\frac{\varepsilon(1-\tau)}{\varepsilon-1}C\right)^{1-\epsilon}\left[1 - \int_{s}^{S}\frac{1}{\sigma}\phi\left(\frac{p+\pi}{\sigma}\right)dp\right]\right], \tag{A19}$$

The first two equations implicitly define the functions $\mathrm{s}(C,\tau)$ and $\mathrm{S}(C,\tau)$. A simple closed-form solution for these limits exists if $\varepsilon = 2$. For $\varepsilon = 3$ and $\varepsilon = 4$ a more cumbersome closed-form solution exists. Beyond that we have not found any closed

[39]To see this consider the function $\Delta\Pi(p)(e^{p})^{\varepsilon}$. It is positive at $p = p^*$, negative at $p = 0$ and at $p \to \infty$, continuous and concave for positive p. Thus it has 2 roots.

form solution. Plugging those two ex- or implicit functions into the last equation, we arrive at the Phillips curve in Proposition 1. That is, we have compressed equations (A9)- (A15) into one single equation relating inflation and output – the Phillips curve:

$$1 = \left[\int_{\mathsf{s}(C,\tau)}^{\mathsf{S}(C,\tau)} e^{p(1-\epsilon)} \frac{1}{\sigma} \phi \left(\frac{p+\pi}{\sigma} \right) dp + \left(\frac{\varepsilon (1-\tau)}{\varepsilon - 1} C \right)^{1-\epsilon} \left[1 - \int_{\mathsf{s}(C,\tau)}^{\mathsf{S}(C,\tau)} \frac{1}{\sigma} \phi \left(\frac{p+\pi}{\sigma} \right) dp \right] \right] \tag{A20}$$

Note that the terms C and τ only appear as the product $C(1-\tau)$ in this expression. Thus, if we express this Phillips curve in terms of the log of C, $\pi(\log(C);\tau)$ changes in τ lead to parallel horizontal shifts. To see this, consider a particular combination of $\bar{\pi}, \bar{C}, \bar{\tau}$ satisfying the above Phillips curve. Now consider a different level of τ such that $(1-\tau) = (1-\bar{\tau})x$. To satisfy the above equation for $\pi = \bar{\pi}$, it must be that $C = \bar{C}/x$. Thus $\log(C) = \log(\bar{C}) - \log(x)$. This is a parallel horizontal shift of the function $\pi(\log(C);\tau)$.

Welfare. Finally, we prove Proposition 3. The central bank's objective is given by the household's utility function,

$$U = \log(C) - N \tag{A21}$$

That is, in the simplified model there is no difference between the planner having commitment or not.

Using first the labor-market clearing condition (A16) to eliminate N in the utility function, and then the definition of frequency (A14) to eliminate g^0 and then the distribution (A12) to eliminate g^c, we arrive at the following.

$$\begin{aligned} U \;=\; & \log(C) - C\left(\int_s^S e^{p(-\epsilon)} \frac{1}{\sigma} \phi \left(\frac{p+\pi}{\sigma} \right) dp + \left(1 - \int_s^S \frac{1}{\sigma} \phi \left(\frac{p+\pi}{\sigma} \right) dp \right) e^{p^*(-\epsilon)} \right) \\ & -\eta \left(1 - \int_s^S \frac{1}{\sigma} \phi \left(\frac{p+\pi}{\sigma} \right) dp \right) \end{aligned} \tag{A22}$$

Using the firms' reset price (A9) to eliminate w in the Ss conditions (A10), (A11) we get:

$$e^{p^*(1-\epsilon)} - \frac{\epsilon - 1}{\epsilon} e^{p^*(1-\epsilon)} - \eta = e^{s(1-\varepsilon)} - \frac{\epsilon - 1}{\epsilon} e^{s(1-\epsilon)}, \tag{A23}$$

$$e^{p^*(1-\epsilon)} - \frac{\epsilon - 1}{\epsilon} e^{p^*(1-\epsilon)} - \eta = e^{S(1-\varepsilon)} - \frac{\epsilon - 1}{\epsilon} e^{S(1-\epsilon)}, \tag{A24}$$

Equations (A23), (A24), and the definition of the price level (A15) together implicitly define functions $\mathsf{s}(\pi)$, $\mathsf{S}(\pi)$ and $\mathsf{p}^*(\pi)$. Plugging these into the welfare function (A22) we arrive at the expression in the text:

$$
\begin{aligned}
U &= \log(C) - C\left(\int_{s(\pi)}^{S(\pi)} e^{(p)(-\epsilon)}\frac{1}{\sigma}\phi\left(\frac{p+\pi}{\sigma}\right)dp + \left(1-\int_{s(\pi)}^{S(\pi)}\frac{1}{\sigma}\phi\left(\frac{p+\pi}{\sigma}\right)dx\right)e^{p^*(\pi)(-\epsilon)}\right) \\
&\quad -\eta\left[1-\int_{s(\pi)}^{S(\pi)}\frac{1}{\sigma}\phi\left(\frac{p+\pi}{\sigma}\right)dp\right] \qquad \text{(A25)}
\end{aligned}
$$

This welfare function depends only on inflation and consumption. In the Calvo case without idiosyncratic shocks, this representation of the welfare function, when approximated to second order, yields the well-known loss function $-\frac{1}{2}\left[\hat{c}^2 + \epsilon\left(\frac{1-\theta}{\theta}\right)\hat{\pi}^2\right]$ (see Galí 2008) where the 'hat' denotes deviation from the deterministic steady state.

In the menu cost model which we are interested in here, we can decompose the welfare gap relative to the efficient allocation into 3 terms:

$$
\begin{aligned}
U - U^{\text{eff}} &= \underbrace{\log(C) - C - 1}_{\text{Average markup gap}} \\
&\quad \underbrace{-C\left(\int_{s(\pi)}^{S(\pi)} e^{(p)(-\epsilon)}\frac{1}{\sigma}\phi\left(\frac{p+\pi}{\sigma}\right)dp + \left(1-\int_{s(\pi)}^{S(\pi)}\frac{1}{\sigma}\phi\left(\frac{p+\pi}{\sigma}\right)dx\right)e^{p^*(\pi)(-\epsilon)} - 1\right)}_{\text{Markup dispersion}} \\
&\quad \underbrace{-\eta\left(1-\int_{s(\pi)}^{S(\pi)}\frac{1}{\sigma}\phi\left(\frac{p+\pi}{\sigma}\right)dp\right)}_{\text{Adjustment costs}} \\
&= \underbrace{-\log\overline{\mu} - \left(\frac{1}{\overline{\mu}} - 1\right)}_{\text{Average markup}} - \underbrace{\frac{1}{\overline{\mu}}(\zeta^{\mu}-1)}_{\text{Markup dispersion}} - \underbrace{\eta g^0}_{\text{Adjustment costs}}.
\end{aligned}
$$

Appendix B. Welfare decomposition

This appendix proves Proposition 2. It starts by proving three lemmas. The first describes the relationship between output and the average welfare-relevant markup, the second the relationship between price and markup dispersion and the third characterizes the efficient allocation. We also suppress time subscripts for notational convenience.

LEMMA A1. *Let the average welfare-relevant markup* $\overline{\mu} \equiv \left(\int \mu(j)^{1-\epsilon}dj\right)^{\frac{1}{1-\epsilon}}$*, where the welfare-relevant markup is the relative price of firm j divided by its welfare-relevant marginal cost:* $\mu(j) = \frac{P(j)/P}{WRMC(j)}$*, and* $WRMC(j) \equiv wA(j)/A$*. Then in any market equilibrium there is a relationship between average welfare-relevant markup and the output:*

$$
\log Y = \log A - \log\overline{\mu} \qquad \text{(A26)}
$$

or equivalently

$$
Y = \frac{A}{\overline{\mu}} \qquad \text{(A27)}
$$

PROOF. In the proof, we first derive the real welfare-relevant marginal cost and define its "aggregate component" that is common across firms. We show that this aggregate component is what affects the average welfare-relevant markup. Then, we derive an expression for the average efficient markup gap, which proves the lemma.

The real welfare-relevant marginal cost of firm j is

$$WRMC(j) = \frac{\partial\,(wN(j))}{\partial Y(j)} = \frac{wA(j)}{A},$$

where we have used that $N(j) = A(j)Y(j)/A$.

Let the common real welfare-relevant marginal cost $wrmc$ be defined as

$$wrmc \equiv (WRMC(j)/A(j)) = (w/A) = Y/A, \tag{A28}$$

Where we used the labor-market clearing condition eq. (7) and the definition of output $Y = C$ which together ensure that $w = Y$.

The welfare-relevant markup $\mu(j)$ is the relative price divided by the the real welfare-relevant marginal cost:

$$\mu(j) = \frac{P(j)}{P} / \frac{wA(j)}{A} = \frac{P(j)}{A(j)P} / \frac{w}{A} = \frac{e^{p}(j)}{wrmc},$$

where $p(j)$ is the logarithm of the quality-adjusted relative price.

The average welfare-relevant markup $\overline{\mu}$ is

$$\overline{\mu} = \left(\int \mu(j)^{1-\epsilon} dj\right)^{\frac{1}{1-\epsilon}} = \left(\int \frac{e^{p(j)(1-\epsilon)}}{wrmc^{1-\epsilon}} dj\right)^{\frac{1}{1-\epsilon}} = \frac{1}{wrmc}\left(\int e^{p(j)(1-\epsilon)} dj\right)^{\frac{1}{1-\epsilon}} = \frac{1}{wrmc}, \tag{A29}$$

where we used the observation that the average quality-adjusted relative price is one (eq. 6) in equilibrium.

The lemma follows from equations (A28) and (A29). □

We define the complete density $g(p) \equiv g^c(p) + g^0\delta(p)$, which includes both the continuous term $g^c(p)$ defined in equation (20) and the dirac delta $\delta(p)$ times the frequency g^0 defined in equation (21). The second lemma shows the relationship between price dispersion and markup dispersion.

LEMMA A2. *Let the dispersion of the quality-adjusted relative prices be* $\zeta^p \equiv \int e^{p(-\epsilon)} g(p)dp$. *Let the markup dispersion be* $\zeta^\mu \equiv \int (\mu(p)/\overline{\mu})^{-\epsilon} g(\log\mu(p) - \log\overline{\mu})dp$. *Then*

$$\zeta^p = \zeta^\mu. \tag{A30}$$

PROOF.

$$\zeta^p = \int e^{p(-\epsilon)} g(p) dp = \int e^{(\log \mu(p) + \log wrmc)(-\epsilon)} g(\log \mu(p) + wrmc) dp = \int e^{(\log \mu(p) - \log \overline{\mu})(-\epsilon)} g(\log \mu(p) - \log \overline{\mu}) dp = \zeta^{\mu}$$

□

And the third lemma calculates output and labor under the efficient allocation. The lemma implies that the efficient output fluctuates with aggregate productivity but is independent of demand shocks as well as of cost-push shocks.

LEMMA A3. *Let Y^e be the efficient output and N^e be the efficient labor., then*

$$\begin{aligned} N^e &= 1, \\ Y^e &= A. \end{aligned} \tag{A31}$$

PROOF. We obtain the efficient output as the solution to a social planning problem. The problem maximizes household welfare in equation (1) subject to (i) the aggregate consumption equation (3), (ii) aggregate labor supply in ($N_t = \int_i N_t(j)$) and (iii) product-level production functions in (10) with respect to product-level consumption and labor ($C_t(j), N_t(j), j \in [0,1], t = 0, 1, 2, \ldots$).

After some algebra, the optimization problem simplifies to

$$\max_{N_t(j)} \sum_{t=0}^{\infty} \beta^t \log \left[A_t \left(\int N_t(j)^{\frac{\epsilon-1}{\epsilon}} di \right)^{\frac{\epsilon}{\epsilon-1}} \right] - \int N_t(j) di,$$

subject to $\int N_t(j) di = N_t$.

It is straightforward to see that the optimization problem implies that the efficient labor supply is equal across products ($N_t^e(j) = N_t^e$, for all $t = 1, 2, \ldots$). Furthermore, optimality requires that $N_t^e = 1$ for all $t = 1, 2, \ldots$. From this, it is clear that $Y_t^e = A_t N_t^e = A_t$. □

COROLLARY A1. *The efficient product-level consumption ($C^e(j)$) varies across products j inversely proportional to the product-level quality, in particular*

$$C^e(j) = \frac{AN^e}{A(j)}.$$

Under perfect foresight, the efficient real interest rate is implicitly defined by the Euler equation after substituting in efficient consumption:

$$r_t^e = -\log \beta - (1 - \rho_A) \log A_t$$

With Lemmas A1, A2, and A3, we are ready to prove Proposition 2. It is repeated here for convenience.

PROPOSITION A1. *Let $U - U^e$ be the central bank's utility gap relative to the utility under efficient allocation expressed in efficient-consumption-equivalent units. The utility gap can be expressed as a function of the average welfare-relevant markup ($\overline{\mu}$), the markup dispersion (ζ^{μ}), and price adjustment costs as*

$$U - U^e = \underbrace{\underbrace{-\log\overline{\mu} - \left(\frac{1}{\overline{\mu}} - 1\right)}_{\text{Average markup}} - \underbrace{\frac{1}{\overline{\mu}}\underbrace{(\zeta^{\mu} - 1)}}_{\text{Markup dispersion}}}_{\text{Misallocation}} - \underbrace{\eta g^0}_{\text{Adjustment costs}}, \tag{A32}$$

where the markup dispersion is $\zeta^{\mu} \equiv \int (\mu(j)/\overline{\mu})^{-\epsilon} g(\log\mu(j) - \log\overline{\mu})dj$, and ηg^0 are the price adjustment costs in labor units.

PROOF. We can express the difference between utility (U) from the utility in the efficient equilibrium (U^e) as

$$\begin{aligned} U - U^e &= (\log C - N) - (\log C^e - N^e) \\ &= (\log Y - N) - (\log A - 1) \\ &= -\log\overline{\mu} - \frac{Y}{A}\int e^{p(-\epsilon)} g(p)\, dp - \eta g^0 + 1 \\ &= -\log\overline{\mu} - \left(\frac{1}{\overline{\mu}}\zeta^{\mu}_t - 1\right) - \eta g^0 \\ &= -\log\overline{\mu} - \left(\frac{1}{\overline{\mu}} - 1\right) - \frac{1}{\overline{\mu}}(\zeta^{\mu} - 1) - \eta g^0 \end{aligned}$$

where U^e is the utility in the efficient equilibrium. The first step of the derivation uses the definition of output $Y = C$ and the efficient output $Y^e = A$. The second step uses Lemma A1 and the labor market equilibrium (23). The third step uses Lemma A2. □

Appendix C. Response to TFP shocks

This appendix proves that, in response to a TFP shock, optimal timeless commitment policy keeps inflation at its steady-state level $\pi_t = \pi$.

The central bank's problem is:

$$\max_{\substack{\{g^c_t(\cdot), g^0_t, V_t(\cdot), C_t, N_t, \\ w_t, p^*_t, s_t, S_t, \pi^*_t\}_{t=0}^{\infty}}} \sum_{t=0}^{\infty} \beta^t (\log C_t - N_t)$$

subject to

$$\begin{aligned} w_t &= C_t, \\ N_t &= \frac{C_t}{A_t}\left(\int_{s_t}^{S_t} e^{(x+p^*_t)(-\epsilon_t)} g^c_t(p)\, dx + g^0_t e^{(p^*_t)(-\epsilon)}\right) - \eta g^0_t \end{aligned}$$

$$
\begin{aligned}
V_t(x) &= \Pi_t(x) + \frac{\Lambda_{t,t+1}}{\sigma}\int_{s_{t+1}}^{S_{t+1}}\left[V_{t+1}(x')\phi\left(\frac{(x-x')-\pi^*_{t+1}}{\sigma}\right)\right]dx' + \\
&\quad \Lambda_{t,t+1}\left(1-\frac{1}{\sigma}\int_{s_{t+1}}^{S_{t+1}}\left[\phi\left(\frac{(x-x')-\pi^*_{t+1}}{\sigma}\right)\right]dx'\right)\left[\left(V_{t+1}(0)-\eta w_{t+1}\right)\right], \\
V_t(s_t) &= V_t(0)-\eta w_t, \\
V_t(S_t) &= V_t(0)-\eta w_t, \\
0 &= \Pi'_t(0) + \frac{\Lambda_{t,t+1}}{\sigma}\int_{s_{t+1}}^{S_{t+1}} V_{t+1}(x')\left.\frac{\partial\phi\left(\frac{x-x'-\pi^*_{t+1}}{\sigma}\right)}{\partial x}\right|_{x=0} dx' \\
&\quad +\frac{\Lambda_{t,t+1}}{\sigma}\left(\phi\left(\frac{-S_{t+1}-\pi^*_{t+1}}{\sigma}\right)-\phi\left(\frac{-s_{t+1}-\pi^*_{t+1}}{\sigma}\right)\right)\left(V_{t+1}(0)-\eta w_{t+1}\right). \\
g^c_t(x) &= \frac{1}{\sigma}\int_{s_{t-1}}^{S_{t-1}} g^c_{t-1}(x_{-1})\phi\left(\frac{x_{-1}-x-\pi^*_t}{\sigma}\right)dx_{-1} + g^0_{t-1}\phi\left(\frac{-x-\pi^*_t}{\sigma}\right), \\
g^0_t &= 1-\int_{s_t}^{S_t} g^c_t(x)dx, \\
1 &= \int_{s_t}^{S_t} e^{\left(x+p^*_t\right)(1-\epsilon)} g^c_t(x)\,dx + g^0_t e^{\left(p^*_t\right)(1-\epsilon)}.
\end{aligned}
$$

We now transform it in a convenient fashion. First, normalize the constraints involving $V_t(x)$ by A_t and substitute for the wage $w_t = C_t$ and the discount factor $\Lambda_{t,t+1} = \beta\frac{C_t}{C_{t+1}}$. With this, the constrains involving $V_t(x)$ become:

$$
\begin{aligned}
\frac{V_t(x)}{A_t} &= \frac{C_t}{A_t}\left(\exp\left(x_t+p^*_t\right)\right)^{1-\epsilon} - \frac{C_t}{A_t}(1-\tau_t)\frac{C_t}{A_t}\left(\exp\left(x_t+p^*_t\right)\right)^{-\epsilon} \\
&\quad +\beta\frac{A_{t+1}}{A_t}\frac{C_t}{C_{t+1}}\frac{1}{\sigma}\int_{s_{t+1}}^{S_{t+1}}\left[\frac{V_{t+1}(x')}{A_{t+1}}\phi\left(\frac{(x-x')-\pi^*_{t+1}}{\sigma}\right)\right]dx' \\
&\quad +\frac{A_{t+1}}{A_t}\beta\frac{C_t}{C_{t+1}}\left(1-\frac{1}{\sigma}\int_{s_{t+1}}^{S_{t+1}}\left[\phi\left(\frac{(x-x')-\pi^*_{t+1}}{\sigma}\right)\right]dx'\right)\left[\left(\frac{V_{t+1}(0)}{A_{t+1}}-\eta\frac{C_{t+1}}{A_{t+1}}\right)\right], \\
\frac{V_t(s_t)}{A_t} &= \frac{V_t(0)}{A_t}-\eta\frac{C_t}{A_t}, \\
\frac{V_t(S_t)}{A_t} &= \frac{V_t(0)}{A_t}-\eta\frac{C_t}{A_t}, \\
0 &= (1-\epsilon)\frac{C_t}{A_t}\left(\exp\left(x_t+p^*_t\right)\right)^{1-\epsilon} + \epsilon\frac{C_t}{A_t}(1-\tau_t)\frac{C_t}{A_t}\left(\exp\left(x_t+p^*_t\right)\right)^{-\epsilon} \\
&\quad +\frac{1}{\sigma}\beta\frac{A_{t+1}}{A_t}\frac{C_t}{C_{t+1}}\int_{s_{t+1}}^{S_{t+1}}\frac{V_{t+1}(x')}{A_{t+1}}\left.\frac{\partial\phi\left(\frac{x-x'-\pi^*_{t+1}}{\sigma}\right)}{\partial x}\right|_{x=0} dx' \\
&\quad +\frac{1}{\sigma}\beta\frac{A_{t+1}}{A_t}\frac{C_t}{C_{t+1}}\left(\phi\left(\frac{-S_{t+1}-\pi^*_{t+1}}{\sigma}\right)-\phi\left(\frac{-s_{t+1}-\pi^*_{t+1}}{\sigma}\right)\right)\left(\frac{V_{t+1}(0)}{A_{t+1}}-\eta\frac{C_{t+1}}{A_{t+1}}\right).
\end{aligned}
$$

Second, define $\frac{V_t(x)}{A_t} \equiv \hat{V}_t(x)$, so that these constrains become

$$\begin{aligned}
\hat{V}_t(x) &= \frac{C_t}{A_t}\left(\exp\left(x_t + p_t^*\right)\right)^{1-\epsilon} - \frac{C_t}{A_t}(1-\tau_t)\frac{C_t}{A_t}\left(\exp\left(x_t + p_t^*\right)\right)^{-\epsilon} \\
&\quad + \beta\frac{C_t}{A_t}\frac{A_{t+1}}{C_{t+1}}\frac{1}{\sigma}\int_{s_{t+1}}^{S_{t+1}}\left[\hat{V}_{t+1}(x')\phi\left(\frac{(x-x') - \pi_{t+1}^*}{\sigma}\right)\right]dx' \\
&\quad + \beta\frac{C_t}{A_t}\frac{A_{t+1}}{C_{t+1}}\left(1 - \frac{1}{\sigma}\int_{s_{t+1}}^{S_{t+1}}\left[\phi\left(\frac{(x-x') - \pi_{t+1}^*}{\sigma}\right)\right]dx'\right)\left[\left(\hat{V}_{t+1}(0) - \eta\frac{C_{t+1}}{A_{t+1}}\right)\right], \\
\hat{V}_t(s_t) &= \hat{V}_t(0) - \eta\frac{C_t}{A_t}, \\
\hat{V}_t(S_t) &= \hat{V}_t(0) - \eta\frac{C_t}{A_t}, \\
0 = \hat{V}_t'(0) &= (1-\epsilon)\frac{C_t}{A_t}\left(\exp\left(x_t + p_t^*\right)\right)^{1-\epsilon} + \epsilon\frac{C_t}{A_t}(1-\tau_t)\frac{C_t}{A_t}\left(\exp\left(x_t + p_t^*\right)\right)^{-\epsilon} \\
&\quad + \frac{1}{\sigma}\beta\frac{C_t}{A_t}\frac{A_{t+1}}{C_{t+1}}\int_{s_{t+1}}^{S_{t+1}}\hat{V}_{t+1}(x')\left.\frac{\partial\phi\left(\frac{x-x'-\pi_{t+1}^*}{\sigma}\right)}{\partial x}\right|_{x=0} dx' \\
&\quad + \frac{1}{\sigma}\beta\frac{C_t}{A_t}\frac{A_{t+1}}{C_{t+1}}\left(\phi\left(\frac{-S_{t+1} - \pi_{t+1}^*}{\sigma}\right) - \phi\left(\frac{-s_{t+1} - \pi_{t+1}^*}{\sigma}\right)\right)\left(\hat{V}(0) - \eta\frac{C_{t+1}}{A_{t+1}}\right).
\end{aligned}$$

Finally, define $\hat{C}_t = \frac{C_t}{A_t}$. The central bank's problem becomes

$$\max_{\substack{\{g_t^c(\cdot), g_t^0, \hat{V}_t(\cdot), \hat{C}_t, \\ w_t, p_t^*, s_t, S_t, \pi_t^*, L_t\}_{t=0}^{\infty}}} \sum_{t=0}^{\infty}\beta^t\left(\log\left(\hat{C}\right) + \log\left(A_t\right) - L_t\right)$$

$$\begin{aligned}
\hat{V}_t(x) &= \hat{C}_t\left(\exp\left(x_t + p_t^*\right)\right)^{1-\epsilon} - \hat{C}_t(1-\tau_t)\hat{C}_t\left(\exp\left(x_t + p_t^*\right)\right)^{-\epsilon} \\
&\quad + \beta\hat{C}_t\hat{C}_{t+1}^{-1}\frac{1}{\sigma}\int_{s_{t+1}}^{S_{t+1}}\left[\hat{V}_{t+1}(x')\phi\left(\frac{(x-x') - \pi_{t+1}^*}{\sigma}\right)\right]dx' \\
&\quad + \beta\hat{C}_t\hat{C}_{t+1}^{-1}\left(1 - \frac{1}{\sigma}\int_{s_{t+1}}^{S_{t+1}}\left[\phi\left(\frac{(x-x') - \pi_{t+1}^*}{\sigma}\right)\right]dx'\right)\left[\left(\hat{V}_{t+1}(0) - \eta\hat{C}_{t+1}\right)\right], \\
\hat{V}_t(s_t) &= \hat{V}_t(0) - \eta\hat{C}_t, \\
\hat{V}_t(S_t) &= \hat{V}_t(0) - \eta\hat{C}_t, \\
0 = \hat{V}_t'(0) &= (1-\epsilon)\hat{C}_t\left(\exp\left(x_t + p_t^*\right)\right)^{1-\epsilon} + \epsilon\hat{C}_t(1-\tau_t)\hat{C}_t\left(\exp\left(x_t + p_t^*\right)\right)^{-\epsilon} \\
&\quad + \frac{1}{\sigma}\beta\hat{C}_t\hat{C}_{t+1}^{-1}\int_{s_{t+1}}^{S_{t+1}}\hat{V}_{t+1}(x')\left.\frac{\partial\phi\left(\frac{x-x'-\pi_{t+1}^*}{\sigma}\right)}{\partial x}\right|_{x=0} dx' \\
&\quad + \frac{1}{\sigma}\beta\hat{C}_t\hat{C}_{t+1}^{-1}\left(\phi\left(\frac{-S_{t+1} - \pi_{t+1}^*}{\sigma}\right) - \phi\left(\frac{-s_{t+1} - \pi_{t+1}^*}{\sigma}\right)\right)\left(\hat{V}(0) - \eta\hat{C}_{t+1}\right).
\end{aligned}$$

$$
\begin{aligned}
N_t &= \hat{C}_t\left(\int_{s_t}^{S_t} e^{(x+p_t^*)(-\epsilon_t)} g_t^c\,(p)\,dx + g_t^0 e^{(p_t^*)(-\epsilon)}\right)\\
g_t^c(x) &= \frac{1}{\sigma}\int_{s_{t-1}}^{S_{t-1}} g_{t-1}^c(x_{-1})\phi\left(\frac{x_{-1}-x-\pi_t^*}{\sigma}\right)dx_{-1} + g_{t-1}^0\phi\left(\frac{-x-\pi_t^*}{\sigma}\right),\\
g_t^0 &= 1-\int_{s_t}^{S_t} g_t^c(x)dx,\\
1 &= \int_{s_t}^{S_t} e^{(x+p_t^*)(1-\epsilon)} g_t^c\,(x)\,dx + g_t^0 e^{(p_t^*)(1-\epsilon)}.
\end{aligned}
$$

Notice that TFP A_t only appears in the objective in a separable way. Therefore, the redefined Ramsey policy is independent of TFP shocks. Going back to the original variables definition, this implies that under optimal policy $C_t \propto A_t$ and $V_t(x) \propto A_t$ while all other variables remain constant at their steady-state values. Thus, inflation π_t also remains constant at its steady-state value.

Appendix D. Optimality condition of the reset price

If the *post decision* value function $V(\cdot)$ is convex, the optimal reset price is fully characterized by the system of first-order conditions in Section 2.2.[40] This appendix presents the derivation of $V_t'(0)$.

To start, we reproduce the value function presented in equation (17), which we then rewrite using $\Phi(\cdot)$ to denote the standard normal c.d.f.

$$
\begin{aligned}
V_t(x) &= \Pi_t(x) + \frac{\Lambda_{t,t+1}}{\sigma}\int_{s_{t+1}}^{S_{t+1}}\left[V_{t+1}(x')\phi\left(\frac{x-x'-\pi_{t+1}^*}{\sigma}\right)\right]dx'\\
&\quad+\Lambda_{t,t+1}\left(1-\frac{1}{\sigma}\int_{s_{t+1}}^{S_{t+1}}\left[\phi\left(\frac{x-x'-\pi_{t+1}^*}{\sigma}\right)\right]dx'\right)\left(V_{t+1}(0)-\eta w_{t+1}\right)\\
V_t(x) &= \Pi_t(x) + \frac{\Lambda_{t,t+1}}{\sigma}\int_{s_{t+1}}^{S_{t+1}}\left[V_{t+1}(x')\phi\left(\frac{x-x'-\pi_{t+1}^*}{\sigma}\right)\right]dx'\\
&\quad+\Lambda_{t,t+1}\left(1-\left[\Phi\left(\frac{x-s_{t+1}-\pi_{t+1}^*}{\sigma}\right)-\Phi\left(\frac{x-S_{t+1}-\pi_{t+1}^*}{\sigma}\right)\right]\right)\left(V_{t+1}(0)-\eta w_{t+1}\right)
\end{aligned}
$$

Taking the derivative of $V_t(x)$ with respect to x and reformulating, we get $V_t'(x)$:

$$
\begin{aligned}
V_t'(x) &= \Pi_t'(x) + \frac{\Lambda_{t,t+1}}{\sigma}\frac{\partial\int_{s_{t+1}}^{S_{t+1}}V_{t+1}(x')\phi\left(\frac{x-x'-\pi_{t+1}^*}{\sigma}\right)dx'}{\partial x}\\
&\quad+\frac{\Lambda_{t,t+1}}{\sigma}\left(\phi\left(\frac{x-S_{t+1}-\pi_{t+1}^*}{\sigma}\right)-\phi\left(\frac{x-s_{t+1}-\pi_{t+1}^*}{\sigma}\right)\right)\left(V_{t+1}(0)-\kappa w_{t+1}\right)\\
&= \Pi_t'(x) + \frac{\Lambda_{t,t+1}}{\sigma}\int_{s_{t+1}}^{S_{t+1}}V_{t+1}(x')\frac{\partial\phi\left(\frac{x-x'-\pi_{t+1}^*}{\sigma}\right)}{\partial x}dx'\\
&\quad+\frac{\Lambda_{t,t+1}}{\sigma}\left(\phi\left(\frac{x-S_{t+1}-\pi_{t+1}^*}{\sigma}\right)-\phi\left(\frac{x-s_{t+1}-\pi_{t+1}^*}{\sigma}\right)\right)\left(V_{t+1}(0)-\kappa w_{t+1}\right)
\end{aligned}
$$

[40]We verify convexity ex post.

which must be evaluated at $x = 0$.

Appendix E. Comparison of static models and full model

This table compares the static Calvo model, the static Golosov-Lucas model and the dynamic Golosov-Lucas model equation by equation. Note that the support of the distribution and value functions in the dynamic model is x, while it is p in the static model, where $p = p^* + x$.

Expression	Static Calvo	Static Golosov-Lucas	Dynamic Golosov-Lucas
Labor supply	$C = w$	$C = w$	$C = w$
Price level	$1 = (1-\theta)\left(\frac{1}{e^{\pi}}\right)^{1-\epsilon} + \theta\, p^*$	$1 = \int_s^S e^{p(1-\epsilon)} g^c(p;\pi)\, dp + g^0 e^{p^*(1-\epsilon)}$	$1 = \int_s^S e^{(x+p_t^*)(1-\varepsilon)} g_t(x)\, dx$
Price star	$p^* = \frac{\epsilon}{\epsilon-1}(1-\tau)w$	$p^* = \frac{\epsilon}{\epsilon-1}(1-\tau)w$	$0 = \Pi_t'(0) + \frac{\Lambda_{t,t+1}}{\sigma} \int_{s_{t+1}}^{S_{t+1}} V_{t+1}(x') \frac{\partial\phi\left(\frac{x-x'-\pi_{t+1}^*}{\sigma}\right)}{\partial x}\, dx'$ $+ \frac{\Lambda_{t,t+1}}{\sigma}\left(\phi\left(\frac{-S_{t+1}-\pi_{t+1}^*}{\sigma}\right) - \phi\left(\frac{-s_{t+1}-\pi_{t+1}^*}{\sigma}\right)\right)\left(V_{t+1}(0) - \eta\, w_{t+1}\right)$
Labor clearing	$N = C\left[\theta\,(p^*)^{-\varepsilon} + (1-\theta)\left(\frac{1}{e^{\pi}}\right)^{-\varepsilon}\right]$	$N = C\left(\int_{-\infty}^{+\infty} e^{p(-\epsilon)} g^c(p;\pi)\, dp + g^0 e^{p^*(-\epsilon)}\right) + \eta\, g^0$	$N = \frac{C_t}{A_t}\left(\int_{s_t}^{S_t} e^{(x+p_t^*)(-\epsilon_t)} g_t^c(p)\, dx + g_t^0 e^{p_t^*(-\epsilon)}\right) - \eta\, g_t^0$
Lower bound	-	$(p^*)^{1-\varepsilon} - (1-\tau)w\,(p^*)^{-\varepsilon} C - \eta\, C$ $= s^{1-\varepsilon} - (1-\tau)w\, s^{-\varepsilon} C$	$V_t(0) - \eta\, w_t = V_t(s_t)$
Upper bound	-	$(p^*)^{1-\varepsilon} - (1-\tau)w\,(p^*)^{-\varepsilon} C - \eta\, C$ $= S^{1-\varepsilon} - (1-\tau)w\, s^{-\varepsilon} C$	$V_t(0) - \eta\, w_t = V_t(S_t)$
Price change freq	$g^0 = \theta$	$g^0 = 1 - \int_s^S g^c(p+\pi)\, dp$	$g_t^0 = 1 - \int_{s_t}^{S_t} g_t^c(x)\, dx$
Price gap density	-	-	$g_t^c(x) = \frac{1}{\sigma}\int_{s_{t-1}}^{S_{t-1}} g_{t-1}^c(x_{-1})\phi\left(\frac{x_{-1} - x - \pi_t^*}{\sigma}\right) dx_{-1} + g_{t-1}^0\, \phi\left(\frac{-x-\pi_t^*}{\sigma}\right)$
Bellman equation	-	-	$V_t(x) = \Pi(x, p_t^*, w_t, A_t)$ $+ \frac{\Lambda_{t,t+1}}{\sigma} \int_{s_{t+1}}^{S_{t+1}} V_{t+1}(x')\phi\left(\frac{(x-x') - \pi_{t+1}^*}{\sigma}\right) dx'$ $+ \Lambda_{t,t+1}\left(1 - \frac{1}{\sigma}\int_{s_{t+1}}^{S_{t+1}} \phi\left(\frac{(x-x') - \pi_{t+1}^*}{\sigma}\right) dx'\right)$ $\times(V_{t+1}(0) - \eta\, w_{t+1})$
Variable count	6	8	10
Equation count	5	7	9

TABLE A1. Comparison of Different Models

Appendix F. Computational algorithm

This appendix explains the computational method. We use a three-step approach to convert the original infinite-dimensional Ramsey problem into a finite-dimensional one. First, we approximate the distribution and value functions by piece-wise linear functions over a set of nodes. Second, we use endogenous nodes, such that both bounds of the (s_t, S_t) band and the optimal reset price are "on the grid". Third, given this approximation, we evaluate integrals analytically. Step one makes the problem finite dimensional. Steps two and three ensure that the approximation is accurate, smooth and computationally efficient. We explain those steps in detail below.

Once we have converted the central bank's infinite-dimensional problem into a finite-dimension problem in this way, we derive the central bank's first order conditions. For this we use symbolic differentiation, and in particular, Dynare's `Ramsey` command. The resulting set of first order conditions is then solved in the sequence space under perfect foresight. Here we employ a standard Newton method using Dynare's `perfect foresight solver` command.

To determine the appropriate initial and terminal conditions, and an initial guess for the transition paths, we need to find the non-stochastic steady state of the model. We determine the steady state of the private equilibrium conditional on a particular value of the policy instrument π using a standard Newton based solution method. We then use this function and exploit the linearity of the first order conditions with respect to the Lagrange multipliers to convert the high-dimensional problem of solving for the steady state into a one-dimensional problem, which is solved with a Newton solver. This last step is performed by Dynare's `steady` command. That is, we manually convert the problem into a finite-dimension problem and find the steady state conditional on a policy; the rest of the procedure uses Dynare.

The rest of the appendix explains those steps that are not straightforward applications of existing methods. It is organized as follows. First, we explain how to make the central bank's problem finite dimensional. For this purpose, we first define some useful auxiliary functions in Section F.1. Then we transform the equilibrium conditions to apply an endogenous grid and approximate the value and distribution functions by a piece-wise linear function in Section F.2. Finally, we evaluate the integrals analytically in Section F.3. The result is a discrete set of equations that can conveniently be represented in matrix form, which we summarize in Section F.4. Second, we explain how we determine the steady state in Section F.5.

F.1. Preliminaries

To begin with, let us normalize the variable x_t as

$$\mathrm{x}_t = \begin{cases} \frac{x_t}{s_t} & \text{if } x_t < 0 \\ \frac{x_t}{S_t} & \text{otherwise} \end{cases} \tag{A33}$$

Under this normalization, the optimal price is at $\mathrm{x}_t = 0$, the upper limit of the (S, s) band at $\mathrm{x}_t = 1$ and the lower limit of the (S, s) band at $\mathrm{x}_t = -1$. This will later allow us to have all critical points (s_t, S_t, p_t^*) on the grid. The law of motion of x_t conditional on not updating can be derived from $x_t = x_{t-1} - \sigma\varepsilon_t - \pi_t^*$:

$$\mathrm{x}_t = \begin{cases} \frac{x_t}{S_t} = \frac{x_{t-1}-\sigma\varepsilon_t-\pi_t^*}{S_t} = \begin{cases} \frac{x_{t-1}-\sigma\varepsilon_t-\pi_t^*}{S_{t-1}} \frac{S_{t-1}}{S_t} = \mathrm{x}_{t-1}\frac{S_{t-1}}{S_t} - \frac{\sigma\varepsilon_t+\pi_t^*}{S_t} & \text{if } \mathrm{x}_t > 0, \text{if } \mathrm{x}_{t-1} > 0 \\ \frac{x_{t-1}-\sigma\varepsilon_t-\pi_t^*}{s_{t-1}} \frac{s_{t-1}}{S_t} = \mathrm{x}_{t-1}\frac{s_{t-1}}{S_t} - \frac{\sigma\varepsilon_t+\pi_t^*}{S_t} & \text{if } \mathrm{x}_t > 0, \text{if } \mathrm{x}_{t-1} < 0 \end{cases} \\ \frac{x_t}{s_t} = \frac{x_{t-1}-\sigma\varepsilon_t-\pi_t^*}{s_t} = \begin{cases} \frac{x_{t-1}-\sigma\varepsilon_t-\pi_t^*}{S_{t-1}} \frac{S_{t-1}}{s_t} = \mathrm{x}_{t-1}\frac{S_{t-1}}{s_t} - \frac{\sigma\varepsilon_t+\pi_t^*}{s_t} & \text{if } \mathrm{x}_t < 0, \text{if } \mathrm{x}_{t-1} > 0 \\ \frac{x_{t-1}-\sigma\varepsilon_t-\pi_t^*}{s_{t-1}} \frac{s_{t-1}}{s_t} = \mathrm{x}_{t-1}\frac{s_{t-1}}{s_t} - \frac{\sigma\varepsilon_t+\pi_t^*}{s_t} & \text{if } \mathrm{x}_t < 0, \text{if } \mathrm{x}_{t-1} < 0 \end{cases} \end{cases} \tag{A34}$$

We now define functions to be used in the next sections to redefine the value and distribution functions. For compactness, let us adopt the notation where $\hat{s}_t(\mathrm{x}_t)$ picks the respective extremes (S, s) depending on the value of x_t following (A33). For brevity, at times we will drop the dependence on x_t and just write $\hat{s}_t$.

Solving (A34) for x_t, x_{t-1} and ε respectively, we obtain the following relations:

$$\mathrm{x}_t = \mathrm{x}_{t-1}\frac{\hat{s}_{t-1}}{\hat{s}_t} - \frac{\sigma\varepsilon_t + \pi_t^*}{\hat{s}_t} \tag{A35}$$

$$\mathrm{x}_{t-1} = \mathrm{x}_t\frac{\hat{s}_t}{\hat{s}_{t-1}} + \frac{\sigma\varepsilon_t + \pi_t^*}{\hat{s}_{t-1}} \tag{A36}$$

$$\varepsilon_t = \frac{\hat{s}_{t-1}\mathrm{x}_{t-1} - \hat{s}_t\mathrm{x}_t - \pi_t^*}{\sigma} \equiv h(\mathrm{x}_{t-1}, \mathrm{x}_t) \tag{A37}$$

where we have defined $h(\mathrm{x}_{t-1}, \mathrm{x}_t)$ for later use.

F.2. Approximating the distribution and value functions by piecewise linear functions on an endogenous grid

Now we redefine the value and distribution functions over the variable x and approximate them by piece-wise linear functions. The original infinite-dimensional problem of the central bank is laid out in Section 3.1. In the following, we consider each of the equations that contain the distribution and value functions one by one.

F.2.1. Distribution

The distribution function is given by

$$g_t(x) \equiv g_t^c(x) + g_t^0\delta(x).$$

where

$$g_t^c(x) = \begin{cases} \frac{1}{\sigma}\int_{s_{t-1}}^{S_{t-1}} g_{t-1}^c(x_{-1})\phi\left(\frac{x_{-1}-x-\pi_t^*}{\sigma}\right)dx_{-1} + g_{t-1}^0\phi\left(\frac{-x-\pi_t^*}{\sigma}\right), & \text{if } x \in [s_t, S_t], \\ 0, & \text{otherwise,} \end{cases} \tag{A38}$$

$$g_t^0 = 1 - \int_{s_t}^{S_t} g_t^c(x)dx.$$

Now we rewrite the distribution using the newly defined re-normalized x where $x = \mathrm{x}\hat{s}_t$ as in equation (A33): define $g_t^c(\mathrm{x}\hat{s}_t) \equiv \mathrm{g}_t^c(\mathrm{x})$ and write

$$\mathrm{g}_t^c(\mathrm{x}) = \begin{cases} \int_{-1}^{1} \frac{\hat{s}_{t-1}(\mathrm{x}')}{\sigma}\mathrm{g}_{t-1}^c(\mathrm{x}')\phi\left(h(\mathrm{x}',\mathrm{x})\right)d\mathrm{x}' + g_{t-1}^0\phi\left(\frac{-\mathrm{x}-\pi_t^*}{\sigma}\right), & \text{if } \mathrm{x} \in [-1,1], \\ 0, & \text{otherwise,} \end{cases} \tag{A39}$$

$$g_t^0 = 1 - \int_{-1}^{1} \mathrm{g}_t^c(\mathrm{x})\,\hat{s}_t(\mathrm{x})d\mathrm{x}. \tag{A40}$$

To see where this comes from, note that for the latter expression for g_t^0 we have applied a simple change of variable to the integral. In particular, we have used the following substitution:

$$\begin{aligned} \int_{s_t}^{S_t} g_t^c(x)dx &= \int_{s_t}^{S_t} g_t^c(\mathrm{x}\hat{s}_t(\mathrm{x}))d\mathrm{x}\hat{s}_t(\mathrm{x}) \\ &= \int_{s_t}^{S_t} \mathrm{g}_t^c(\mathrm{x})d\mathrm{x}\hat{s}_t(\mathrm{x}) = \int_{s_t/\hat{s}_t(\mathrm{x})}^{S_t/\hat{s}_t(\mathrm{x})} \hat{s}_t(\mathrm{x})\mathrm{g}_t^c(\mathrm{x})d\mathrm{x} = \int_{-1}^{1} \hat{s}_t(\mathrm{x})\mathrm{g}_t^c(\mathrm{x})d\mathrm{x}. \end{aligned}$$

Next, we will also change the variable in the integral in the equation for $\mathrm{g}_t^c(\mathrm{x})$ (A39). This change of variable is a bit more involved. First, we re-express (A38) as

$$\mathrm{g}_t^c(\mathrm{x}) = \begin{cases} \frac{1}{\sigma}\int_{s_{t-1}}^{S_{t-1}} \mathrm{g}_{t-1}^c(\mathrm{x}_{-1})\phi\left(\frac{\mathrm{x}_{-1}\hat{s}_{t-1}-\mathrm{x}\hat{s}_t-\pi_t^*}{\sigma}\right)d(\mathrm{x}_{-1}\hat{s}_{t-1}) + g_{t-1}^0\phi\left(\frac{-\mathrm{x}\hat{s}_t-\pi_t^*}{\sigma}\right), & \text{if } \mathrm{x} \in [-1,1], \\ 0, & \text{otherwise,} \end{cases}$$

Second, we split the integral in two parts at 0 (and we drop the second line of the above expression for brevity)

$$\begin{aligned} \mathrm{g}_t^c(\mathrm{x}) \quad = \quad & \frac{1}{\sigma}\int_{s_{t-1}}^{0} \mathrm{g}_{t-1}^c(\mathrm{x}_{-1})\phi\left(\frac{\mathrm{x}_{-1}s_{t-1} - \mathrm{x}\hat{s}_t - \pi_t^*}{\sigma}\right)d(\mathrm{x}_{-1}s_{t-1}) \\ & + \frac{1}{\sigma}\int_{0}^{S_{t-1}} \mathrm{g}_{t-1}^c(\mathrm{x}_{-1})\phi\left(\frac{\mathrm{x}_{-1}S_{t-1} - \mathrm{x}\hat{s}_t - \pi_t^*}{\sigma}\right)d(\mathrm{x}_{-1}S_{t-1}) \\ & + g_{t-1}^0\phi\left(\frac{-\mathrm{x}\hat{s}_t - \pi_t^*}{\sigma}\right) \quad \text{if } \mathrm{x} \in [-1,1], \end{aligned}$$

Now we do a change of variable: integrate over x_{-1} instead of $\mathrm{x}_{-1}s_{t-1}$

$$\begin{aligned} \mathrm{g}_t^c(\mathrm{x}) &= \int_1^0 \frac{s_{t-1}}{\sigma} \mathrm{g}_{t-1}^c(\mathrm{x}_{-1}) \phi\left(\frac{\mathrm{x}_{-1}s_{t-1} - \mathrm{x}\hat{s}_t - \pi_t^*}{\sigma}\right) d\mathrm{x}_{-1} \\ &+ \int_0^1 \frac{S_{t-1}}{\sigma} \mathrm{g}_{t-1}^c(\mathrm{x}_{-1}) \phi\left(\frac{\mathrm{x}_{-1}S_{t-1} - \mathrm{x}\hat{s}_t - \pi_t^*}{\sigma}\right) d\mathrm{x}_{-1} \\ &+ g_{t-1}^0 \phi\left(\frac{-\mathrm{x}\hat{s}_t - \pi_t^*}{\sigma}\right) \quad \text{if } \mathrm{x} \in [-1, 1], \end{aligned}$$

Finally, pasting the two integrals together again, re-denoting x_{-1} by x' and using $h(\mathrm{x}', \mathrm{x})$ we get expression (A39). This concludes the explanation of the change of variables.

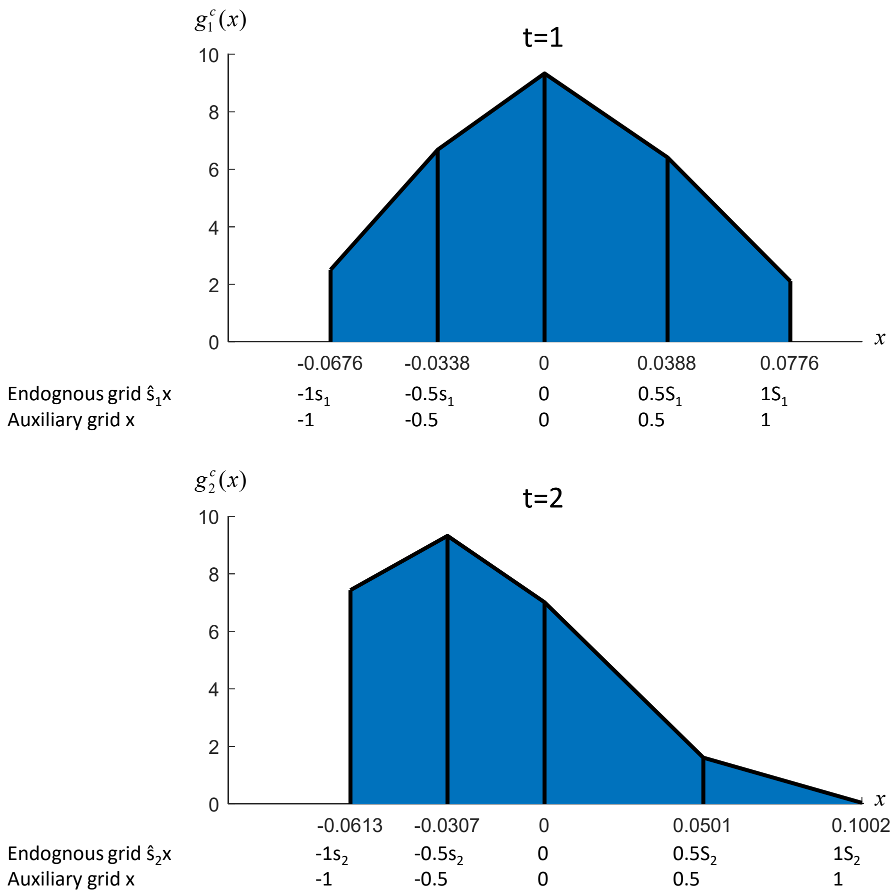

FIGURE A1. This figure schematically explains the linear interpolation with an endogenous grid. It shows the piece-wise linearly approximated distribution $\mathrm{g}_t^c(x)$ at two points in time, $t = 1$ and $t = 2$. The thresholds of the (S, s) band are not symmetric around 0 and differ across time. The endogenous grid x has I grid points, which are automatically adjusted so that half of the grid points cover the negative part of the (s, S) band and half of them cover the positive part. In this illustrative example $I = 5$ (we use a larger I when solving the model). The adjustment is obtained by multiplying the auxiliary grid $\mathrm{x} = [-1, -0.5, 0, 0.5, 1]$ by $\hat{s}_t(\mathrm{x})$: $x = \mathrm{x}\hat{s}_t$

So far we have rewritten the law of motion of the firm distribution g_t. We now introduce the approximation we rely on for g_t. We approximate g^c by a piece-wise linear function with equally spaced nodes $\mathrm{x}_1, \ldots, \mathrm{x}_I = -1, \ldots, 0, \ldots, 1$ with $\mathrm{g}_t^c(\mathrm{x}|\mathrm{x}_i < \mathrm{x} < \mathrm{x}_{i+1}) \approx \mathrm{g}_t^c(\mathrm{x}_i) + \frac{\mathrm{x}-\mathrm{x}_i}{\mathrm{x}_{i+1}-\mathrm{x}_i} \frac{g_{t-1}^c(\mathrm{x}_{i+1})-g_{t-1}^c(\mathrm{x}_i)}{\mathrm{x}_{i+1}-\mathrm{x}_i}$.

Note that the auxiliary grid for x is exogenous. However, this exogenous auxiliary grid defines an *endogenous grid* for $x = \hat{s}_t \mathrm{x}$, which, at each t, exactly spans the (s, S) band and has a node at 0. Figure A1 illustrates the use of linear interpolation with an endogenous grid as we apply it here.

From now on, g_t^c denotes the piece-wise linear approximated function, and $\mathrm{g}_t^c(\mathrm{x}_i < \mathrm{x} < \mathrm{x}_{i+1})$ denotes a linear piece of it. Thus, the functions are approximated as

$$\mathrm{g}_t^c(\mathrm{x}) = \sum_{i=1}^{I-1} \int_{\mathrm{x}_i}^{\mathrm{x}_{i+1}} \frac{\hat{s}_{t-1}(\mathrm{x}')}{\sigma} \mathrm{g}_{t-1}^c(\mathrm{x}_i < \mathrm{x}' < \mathrm{x}_{i+1}) \phi\left(h(\mathrm{x}', \mathrm{x})\right) \, d\mathrm{x}' + \frac{1}{\sigma} \mathrm{g}_{t-1}^0 \phi\left(h(0, \mathrm{x})\right),$$

$$\mathrm{g}_t^0 = 1 - \sum_{i=1}^{I-1} \int_{\mathrm{x}_i}^{\mathrm{x}_{i+1}} \mathrm{g}_t^c(\mathrm{x}_i < \mathrm{x} < \mathrm{x}_{i+1}) \hat{s}_t(\mathrm{x}) d\mathrm{x}.$$

Notice that in these expressions, the integrands are continuous in the interval $\mathrm{x}_i < \mathrm{x} < \mathrm{x}_{i+1}$ since x and x' are of constant sign.

Also note that the distribution function is 0 outside the (S,s) band. Our piecewise linear g_t^c in fact is only defined over the range where the distribution has positive mass, that is, for $\mathrm{x} \in [-1, 1]$. This is computationally efficient.

Within this range = 1 so we can drop it from the expression above.

$$\mathrm{g}_t^c(\mathrm{x}) = \left[\sum_{i=1}^{I-1} \int_{\mathrm{x}_i}^{\mathrm{x}_{i+1}} \frac{\hat{s}_{t-1}(\mathrm{x}')}{\sigma} \mathrm{g}_{t-1}^c(\mathrm{x}_i < \mathrm{x}' < \mathrm{x}_{i+1}) \phi\left(h(\mathrm{x}', \mathrm{x})\right) \, d\mathrm{x}' + \frac{1}{\sigma} \mathrm{g}_{t-1}^0 \phi\left(h(0, \mathrm{x})\right)\right]$$

F.2.2. Other Aggregation Equations

The equilibrium conditions contain two additional aggregation equations that contain the function $\mathrm{g}(\cdot)$, for which we use the piece-wise linear approximation of $\mathrm{g}^c(\cdot)$. Recall the aggregate price index and the labor market clearing condition

$$e^{p_t^*(\epsilon-1)} = \int e^{x(1-\epsilon)} g_t(x) dx,$$

$$N_t = \frac{C_t}{A_t} e^{p_t^*(-\epsilon)} \int e^{x(-\epsilon)} g_t(x) d(x) + \eta \int \lambda_t(x + p_t^* - \sigma\varepsilon_t - \pi_t^*) g_{t-1}(x) d(x)$$

which we approximate as follows, after the change of variable to x,

$$e^{p_t^*(\epsilon-1)} = \sum_{i=1}^{I-1} \int_{\mathrm{x}_i}^{\mathrm{x}_{i+1}} e^{\mathrm{x}(1-\epsilon)} \mathrm{g}_t^c(\mathrm{x}_{i-1} < \mathrm{x} < \mathrm{x}_{i+1}) \hat{s}_t(\mathrm{x}) d\mathrm{x} + \mathrm{g}_t^0,$$

$$N_t = \frac{C_t}{A_t} e^{p^*(-\epsilon)} \sum_{i=1}^{I-1} \int_{\mathrm{x}_i}^{\mathrm{x}_{i+1}} \left(e^{\mathrm{x}(-\epsilon)} \mathrm{g}_t^c(\mathrm{x}_{i-1} < \mathrm{x} < \mathrm{x}_{i+1}) \hat{s}_t(\mathrm{x}) d\mathrm{x} + \mathrm{g}_{t-1}^0\right) + \eta \mathrm{g}_{t-1}^0.$$

F.2.3. Value Function

Recall the value function is

$$
\begin{aligned}
V_t(x) =& \Pi_t(x) + \frac{\Lambda_{t,t+1}}{\sigma} \int_{s_{t+1}}^{S_{t+1}} \left[V_{t+1}(x') \phi\left(\frac{x - x' - \pi^*_{t+1}}{\sigma} \right) \right] dx' \\
&+ \Lambda_{t,t+1} \left(1 - \frac{1}{\sigma} \int_{s_{t+1}}^{S_{t+1}} \left[\phi\left(\frac{x - x' - \pi^*_{t+1}}{\sigma} \right) \right] dx' \right) \left(V_{t+1}(0) - \eta w_{t+1} \right)
\end{aligned}
$$

We now express it in terms of x with $\mathsf{V}_t(\mathsf{x}) \equiv V_t(\mathsf{x}\hat{s}_t)$:

$$
\begin{aligned}
\mathsf{V}_t(\mathsf{x}) =& \Pi_t(\mathsf{x}) + \frac{\Lambda_{t,t+1}}{\sigma} \int_{s_{t+1}}^{S_{t+1}} \left[\mathsf{V}_{t+1}(\mathsf{x}') \phi\left(\frac{\mathsf{x}\hat{s}_t - \mathsf{x}'\hat{s}_{t+1} - \pi^*_{t+1}}{\sigma} \right) \right] d\mathsf{x}'\hat{s}_{t+1} \\
&+ \Lambda_{t,t+1} \left(1 - \frac{1}{\sigma} \int_{s_{t+1}}^{S_{t+1}} \left[\phi\left(\frac{\mathsf{x}\hat{s}_t - \mathsf{x}'\hat{s}_{t+1} - \pi^*_{t+1}}{\sigma} \right) \right] d\mathsf{x}'\hat{s}_{t+1} \right) \left(\mathsf{V}_{t+1}(0) - \eta w_{t+1} \right)
\end{aligned}
$$

Note that, in equilibrium it must hold that $\mathsf{V}_t(0) - \eta \frac{w_{t+1}}{A_{t+1}} = \mathsf{V}_t(-1) = \mathsf{V}_t(1)$ and $\mathsf{V}'_t(0) = 0$. The first two equalities are straightforward; the next subsection discusses the latter.

After the change of variable to x', which is analogous to the change of variable applied to g^c_t previously, we can rewrite $\mathsf{V}_t(\mathsf{x})$ as

$$
\begin{aligned}
\mathsf{V}_t(\mathsf{x}) =& \Pi_t(\mathsf{x}) + \frac{\Lambda_{t,t+1}}{\sigma} \int_{-1}^{1} \left[\hat{s}_{t+1}(\mathsf{x}') \mathsf{V}_{t+1}(\mathsf{x}') \phi\left(h(\mathsf{x}, \mathsf{x}') \right) \right] d\mathsf{x}' \\
&+ \Lambda_{t,t+1} \left(1 - \frac{1}{\sigma} \int_{-1}^{1} \left[\hat{s}_{t+1}(\mathsf{x}') \phi\left(h(\mathsf{x}, \mathsf{x}') \right) \right] d\mathsf{x}' \right) \left(\mathsf{V}_{t+1}(0) - \eta w_{t+1} \right)
\end{aligned}
$$

So far, we have normalized the support of the value function. In addition, it is convenient to normalize the value function itself. We normalize the value function by its maximal value $\mathsf{V}_t(0)$, and denote the normalized value function by $\mathsf{v}_t(\mathsf{x})$: $\mathsf{v}_t(\mathsf{x}) \equiv \mathsf{V}_t(\mathsf{x}) - \mathsf{V}_t(0)$. The expression above can be re-written as:

$$
\begin{aligned}
&\mathsf{v}_t(\mathsf{x}) \equiv \mathsf{V}_t(\mathsf{x}) - \mathsf{V}_t(0) = \Pi_t(\mathsf{x}) - \Pi_t(0) \\
&+ \frac{\Lambda_{t,t+1}}{\sigma} \left(\int_{-1}^{1} \hat{s}_{t+1}(\mathsf{x}') \left[\mathsf{V}_{t+1}(\mathsf{x}') \phi\left(\frac{\mathsf{x} - \mathsf{x}' - \pi^*_{t+1}}{\sigma} \right) - \mathsf{V}_{t+1}(\mathsf{x}') \phi\left(\frac{0 - \mathsf{x}' - \pi^*_{t+1}}{\sigma} \right) \right] d\mathsf{x}' \right) \\
&+ \frac{\Lambda_{t,t+1}}{\sigma} \left(- \int_{-1}^{1} \hat{s}_{t+1}(\mathsf{x}') \left[\phi\left(\frac{\mathsf{x} - \mathsf{x}' - \pi^*_{t+1}}{\sigma} \right) - \phi\left(\frac{0 - \mathsf{x}' - \pi^*_{t+1}}{\sigma} \right) \right] d\mathsf{x}' \right) \left(V_{t+1}(0) - \eta w_{t+1} \right) \\
&= \Pi_t(\mathsf{x}) - \Pi_t(0) \\
&+ \frac{\Lambda_{t,t+1}}{\sigma} \left(\int_{-1}^{1} \hat{s}_{t+1}(\mathsf{x}') \left[\mathsf{v}_{t+1}(\mathsf{x}') \left(\phi\left(\frac{\mathsf{x} - \mathsf{x}' - \pi^*_{t+1}}{\sigma} \right) - \phi\left(\frac{0 - \mathsf{x}' - \pi^*_{t+1}}{\sigma} \right) \right) \right] d\mathsf{x}' \right) \\
&+ \frac{\Lambda_{t,t+1}}{\sigma} \left(- \int_{-1}^{1} \hat{s}_{t+1}(\mathsf{x}') \left[\phi\left(\frac{\mathsf{x} - \mathsf{x}' - \pi^*_{t+1}}{\sigma} \right) - \phi\left(\frac{0 - \mathsf{x}' - \pi^*_{t+1}}{\sigma} \right) \right] d\mathsf{x}' \right) \left(-\eta w_{t+1} \right)
\end{aligned}
$$

Following our approach for $\mathsf{g}^c(\cdot)$, we approximate $\mathsf{v}(\cdot)$ by a piece-wise linear function with nodes $\mathrm{x}_1, \ldots, \mathrm{x}_I = -1, \ldots, 0, \ldots, 1$ with $\mathrm{v}_t(\mathrm{x}|\mathrm{x}_i < \mathrm{x} < \mathrm{x}_{i+1}) \approx \mathrm{v}_t(\mathrm{x}_i) + \frac{\mathrm{x}-\mathrm{x}_i}{\mathrm{x}_{i+1}-\mathrm{x}_i} \frac{\mathrm{v}_t(\mathrm{x}_{i+1})-\mathrm{v}_t(\mathrm{x}_i)}{\mathrm{x}_{i+1}-\mathrm{x}_i}$.

From now on, v_t denotes the piece-wise linear approximated function and $\mathrm{v}_t(\mathrm{x}_i < \mathrm{x} < \mathrm{x}_{i+1})$ denotes a linear piece of it. Thus, this function $\mathrm{v}_t(\mathrm{x})$ is approximated as

$$\begin{aligned}\mathrm{v}_t(\mathrm{x}) =& \Pi_t(\mathrm{x}) - \Pi_t(0)\\ &+ \frac{\Lambda_{t,t+1}}{\sigma} \sum_{i=1}^{I-1} \int_{\mathrm{x}_i}^{\mathrm{x}_{i+1}} \hat{s}_{t+1}(\mathrm{x}')\mathrm{v}_{t+1}(\mathrm{x}_i < \mathrm{x}' < \mathrm{x}_{i+1})(\phi\left(h(\mathrm{x},\mathrm{x}')\right) - \phi\left(h(0,\mathrm{x}')\right))d\mathrm{x}'\\ &+ \frac{\Lambda_{t,t+1}}{\sigma}(-\eta w_{t+1}) \int_{-1}^{1} \hat{s}_{t+1}(\mathrm{x}')(\phi\left(h(\mathrm{x},\mathrm{x}')\right) - \phi\left(h(0,\mathrm{x}')\right))d\mathrm{x}'.\end{aligned}$$

F.2.4. Optimality condition for the reset price

We proceed in the same way for the derivative of the value function. We start with

$$\begin{aligned}0 = V_t'(0) = \Pi_t'(0) + \frac{\Lambda_{t,t+1}}{\sigma} \int_{s_{t+1}}^{S_{t+1}} V_{t+1}(x') \left.\frac{\partial \phi\left(\frac{x-x'-\pi^*_{t+1}}{\sigma}\right)}{\partial x}\right|_{x=0} dx'\\ + \frac{\Lambda_{t,t+1}}{\sigma}\left(\phi\left(\frac{-S_{t+1}-\pi^*_{t+1}}{\sigma}\right) - \phi\left(\frac{-s_{t+1}-\pi^*_{t+1}}{\sigma}\right)\right)(V_{t+1}(0) - \eta w_{t+1})\end{aligned}$$

where

$$\begin{aligned}\left.\frac{\partial \phi\left(\frac{x-x'-\pi^*_{t+1}}{\sigma}\right)}{\partial x}\right|_{x=0} &= \frac{1}{\sqrt{2\pi}\sigma} \frac{-\pi^*_{t+1} - x'}{\sigma} e^{-\frac{1}{2}\left(\frac{-\pi^*_{t+1}-x'}{\sigma}\right)^2},\\ &= \frac{\phi\left(\frac{-\pi^*_{t+1}-x'}{\sigma}\right)}{\sigma} \frac{-\pi^*_{t+1} - x'}{\sigma}\end{aligned}$$

After change of variable to x, this expression becomes

$$\begin{aligned}0 = \Pi_t'(0) + \frac{\Lambda_{t,t+1}}{\sigma} \int_{-1}^{1} \hat{s}_{t+1}(\mathrm{x}')\mathrm{V}_{t+1}(\mathrm{x}')h(0,\mathrm{x}')\frac{\phi\left(h(0,\mathrm{x}')\right)}{\sigma}d\mathrm{x}'\\ + \frac{\Lambda_{t,t+1}}{\sigma}\left(\phi\left(\frac{-S_{t+1}-\pi^*_{t+1}}{\sigma}\right) - \phi\left(\frac{-s_{t+1}-\pi^*_{t+1}}{\sigma}\right)\right)(\mathrm{V}_{t+1}(0) - \eta w_{t+1}).\end{aligned}$$

Now we re-express this in terms of $\mathrm{v}(\mathrm{x})$ using $\mathrm{V}_t(\mathrm{x}) = \mathrm{v}_t(\mathrm{x}) + \mathrm{V}_t(0)$ first, and the rearranging

$$0 = \Pi_t'(x) + \Lambda_{t,t+1} \int_{-1}^{1} \hat{s}_{t+1}(\mathrm{x}')\left(\mathrm{v}_{t+1}(\mathrm{x}') + \mathrm{V}_{t+1}(0)\right)h(0,\mathrm{x}')\frac{\phi\left(h(0,\mathrm{x}')\right)}{\sigma}d\mathrm{x}'$$

$$
\begin{aligned}
&+\frac{\Lambda_{t,t+1}}{\sigma}\left(\phi\left(\frac{-S_{t+1}-\pi^*_{t+1}}{\sigma}\right)-\phi\left(\frac{-s_{t+1}-\pi^*_{t+1}}{\sigma}\right)\right)(\mathrm{V}_{t+1}(0)-\eta w_{t+1})\\
=\;& \Pi'_t(x)+\Lambda_{t,t+1}\int_{-1}^{1}\hat{s}_{t+1}(\mathrm{x}')\mathrm{v}_{t+1}(x')h(0,\mathrm{x}')\frac{\phi\left(h(0,\mathrm{x}')\right)}{\sigma}d\mathrm{x}'\\
&+\Lambda_{t,t+1}\int_{-1}^{1}\hat{s}_{t+1}(\mathrm{x}')h(0,\mathrm{x}')\frac{\phi\left(h(0,\mathrm{x}')\right)}{\sigma}dx'\mathrm{V}_{t+1}(0)\\
&+\frac{\Lambda_{t,t+1}}{\sigma}\left(\phi\left(\frac{-S_{t+1}-\pi^*_{t+1}}{\sigma}\right)-\phi\left(\frac{-s_{t+1}-\pi^*_{t+1}}{\sigma}\right)\right)(\mathrm{V}_{t+1}(0)-\eta w_{t+1})\\
=\;& \Pi'_t(x)+\Lambda_{t,t+1}\int_{-1}^{1}\hat{s}_{t+1}(\mathrm{x}')\mathrm{v}_{t+1}(x')h(0,\mathrm{x}')\frac{\phi\left(h(0,\mathrm{x}')\right)}{\sigma}d\mathrm{x}'\\
&-\frac{\Lambda_{t,t+1}}{\sigma}\left(\phi\left(\frac{-S_{t+1}-\pi^*_{t+1}}{\sigma}\right)-\phi\left(\frac{-s_{t+1}-\pi^*_{t+1}}{\sigma}\right)\right)\mathrm{V}_{t+1}(0)\\
&+\frac{\Lambda_{t,t+1}}{\sigma}\left(\phi\left(\frac{-S_{t+1}-\pi^*_{t+1}}{\sigma}\right)-\phi\left(\frac{-s_{t+1}-\pi^*_{t+1}}{\sigma}\right)\right)(\mathrm{V}_{t+1}(0)-\eta w_{t+1})\\
=\;& \Pi'_t(0)+\Lambda_{t,t+1}\int_{-1}^{1}\hat{s}_{t+1}(\mathrm{x}')\mathrm{v}_{t+1}(\mathrm{x}')h(0,\mathrm{x}')\frac{\phi\left(h(0,\mathrm{x}')\right)}{\sigma}d\mathrm{x}'\\
&+\frac{\Lambda_{t,t+1}}{\sigma}\left(\phi\left(\frac{-S_{t+1}-\pi^*_{t+1}}{\sigma}\right)-\phi\left(\frac{-s_{t+1}-\pi^*_{t+1}}{\sigma}\right)\right)(-\eta w_{t+1})
\end{aligned}
$$

and apply the piece-wise linear approximation of $\mathrm{v}(\mathrm{x})$:

$$
\begin{aligned}
0 =\;& \Pi'_t(0)+\Lambda_{t,t+1}\sum_{i=1}^{I-1}\int_{-1}^{1}\hat{s}_{t+1}(\mathrm{x}')\mathrm{v}_{t+1}(\mathrm{x}_\mathrm{i}<\mathrm{x}'<\mathrm{x}_{\mathrm{i}+1})h(0,\mathrm{x}')\frac{\phi\left(h(0,\mathrm{x}')\right)}{\sigma}d\mathrm{x}'\\
&+\frac{\Lambda_{t,t+1}}{\sigma}\left(\phi\left(\frac{-S_{t+1}-\pi^*_{t+1}}{\sigma}\right)-\phi\left(\frac{-s_{t+1}-\pi^*_{t+1}}{\sigma}\right)\right)(-\eta w_{t+1}).
\end{aligned}
$$

F.3. Solving for Integrals

Let us collect the approximated equations defined so far.

$$
\begin{aligned}
\mathrm{v}_t(\mathrm{x}) =&\Pi_t(\mathrm{x})-\Pi_t(0)\\
&+\frac{\Lambda_{t,t+1}}{\sigma}\sum_{i=1}^{I-1}\int_{\mathrm{x}_i}^{\mathrm{x}_{i+1}}\hat{s}_{t+1}(\mathrm{x}')\mathrm{v}_{t+1}(\mathrm{x}_i<\mathrm{x}'<\mathrm{x}_{i+1})(\phi\left(h(\mathrm{x},\mathrm{x}')\right)-\phi\left(h(0,\mathrm{x}')\right))d\mathrm{x}'\\
&+\frac{\Lambda_{t,t+1}}{\sigma}(-\eta w_{t+1})\int_{-1}^{1}\hat{s}_{t+1}(\mathrm{x}')(\phi\left(h(\mathrm{x},\mathrm{x}')\right)-\phi\left(h(0,\mathrm{x}')\right))d\mathrm{x}',
\end{aligned}
\tag{A41}
$$

$$
\begin{aligned}
0 =\;& \Pi'_t(0)+\Lambda_{t+1}\int_{-1}^{1}\hat{s}_{t+1}\mathrm{v}_{t+1}(\mathrm{x}')h(0,\mathrm{x}')\frac{\phi\left(h(0,\mathrm{x}')\right)}{\sigma}d\mathrm{x}'\\
&+\frac{\Lambda_{t,t+1}}{\sigma}\left(\phi\left(\frac{-S_{t+1}-\pi^*_{t+1}}{\sigma}\right)-\phi\left(\frac{-s_{t+1}-\pi^*_{t+1}}{\sigma}\right)\right)(-\eta w_{t+1}),
\end{aligned}
\tag{A42}
$$

$$
\mathrm{g}^c_t(\mathrm{x}) = \sum_{i=1}^{I-1}\int_{\mathrm{x}_i}^{\mathrm{x}_{i+1}}\frac{\hat{s}_{t-1}(\mathrm{x}')}{\sigma}\mathrm{g}^c_{t-1}(\mathrm{x}_i<\mathrm{x}'<\mathrm{x}_{i+1})\phi\left(h(\mathrm{x}',\mathrm{x})\right)\,d\mathrm{x}'+\frac{1}{\sigma}\mathrm{g}^0_{t-1}\phi\left(h(0,\mathrm{x})\right),
\tag{A43}
$$

$$g_t^0 = 1 - \sum_{i=1}^{I-1} \int_{x_i}^{x_{i+1}} g_t^c(x_i < x' < x_{i+1}) \hat{s}_t(x) dx, \tag{A44}$$

$$e^{p_t^*(\epsilon-1)} = \sum_{i=1}^{I-1} \int_{x_i}^{x_{i+1}} e^{(x)(1-\epsilon)} g_t^c(x_i < x' < x_{i+1}) \hat{s}_t(x) dx + g_t^0, \tag{A45}$$

$$N_t = \frac{C_t}{A_t} e^{p_t^*(-\epsilon)} \left(\sum_{i=1}^{I-1} \int_{x_i}^{x_{i+1}} e^{x(-\epsilon)} g_t^c(x_{i-1} < x < x_{i+1}) \hat{s}_t(x) dx + g_{t-1}^0 \right) + \eta g_{t-1}^0. \tag{A46}$$

The integrals in all of these expressions can be computed analytically, since the integrands consist of affine functions multiplied by expressions that have closed-form anti-derivatives. Figure A2 illustrates this graphically for the integral in the equation for $g_t^c(x)$ (A43).

We now determine the solution of these integrals, equation by equation. Given the coefficients of the affine functions, which depend on the values of $v_{t+1}(g_{t-1})$ at the grid points x_i, we can then write the solutions as a function that is linear in the elements of the vector $v_{t+1}(x_i)$ $(g_{t-1}(x_i))$. We now explain this for the simple case of the integral in equation A44. The other equations require more tedious algebra, which we conveniently executed using symbolic math and which we omit here for brevity, but are conceptually equivalent.

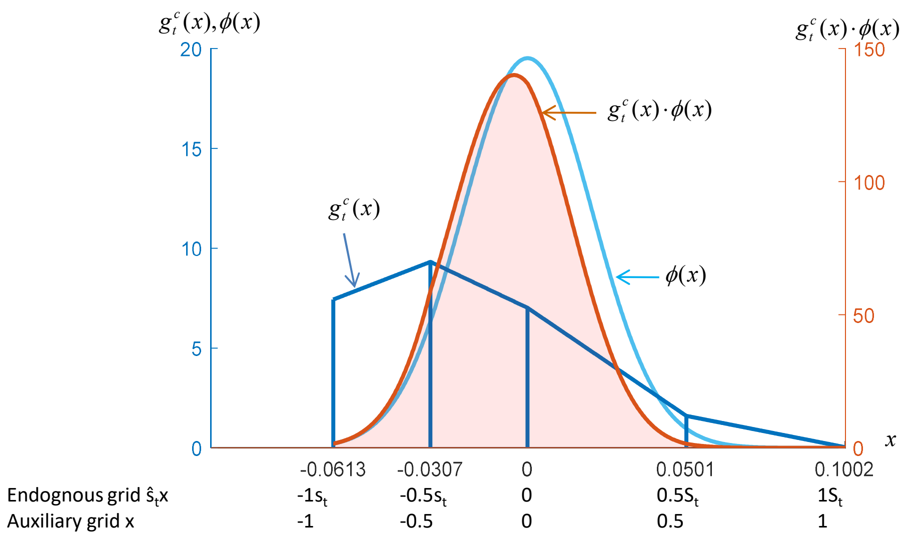

FIGURE A2. This figure schematically explains the analytical evaluation of integrals, given the linear approximation of the distribution and value functions. It shows the piece-wise linearly approximated distribution $g_t^c(x)$ in blue, the normal pdf $\phi(x)$ in light blue and the product of the two $g_t^c(x)\phi(x)$ in orange, where $x = x\hat{s}_t$. The orange area thus corresponds to the term $\sum_{i=1}^{I-1} \int_{x_i}^{x_{i+1}} \frac{\hat{s}_{t-1}(x')}{\sigma} g_{t-1}^c(x_i < x' < x_{i+1}) \phi\left(h(x', x)\right) dx'$ in equation (A43).

F.3.1. Mass Point

The integral over an affine function $f(x)$ from x_1 to x_2 is given by

$$\int_{x_1}^{x_2} f(x)\, dx = \frac{(f(x_1) + f(x_2))}{2}(x_2 - x_1)$$

thus

$$\sum_{i=1}^{I-1} \int_{x_i}^{x_{i+1}} f(x)\, dx = \sum_{i=1}^{I-1} \frac{(f(x_i) + f(x_{i+1}))}{2}(x_{i+1} - x_i).$$

Collecting the common terms on the right-hand side we get

$$\sum_{i=1}^{I-1} \int_{x_i}^{x_{i+1}} f(x)\, dx = \frac{\Delta x}{2}\left(f(x_1) + 2\sum_{i=2}^{I-1} f(x_i) + f(x_I)\right).$$

Applying this formula to equation (A44), which defines the mass point at $x = 0$, and re-arranging terms we get

$$\mathrm{g}_t^0 = 1 - \mathbf{e}_t^T \mathbf{g}_t^{\boldsymbol{c}} \tag{A47}$$

where $\mathbf{e}_t^T = [0.5,\ 1, \ldots, 1,\ 0.5]\Delta x$. Note that this formula corresponds to the trapezoid rule. The blue area in Figure A1 illustrates the application of the trapezoid rule.

F.3.2. Aggregate Price Index

By the same logic, the aggregate price index in (A45) is computed as

$$e^{p_t^*(\epsilon-1)} = \sum_{i=1}^{I}(\mathrm{g}_t^c(\mathrm{x}_i)\mathbb{1}_{i\neq 1}d_{t,i,i-1,1-\varepsilon} + \mathrm{g}_t^c(\mathrm{x}_i)\mathbb{1}_{i\neq I}d_{t,i,i+1,1-\varepsilon}) + \mathrm{g}_t^0 \tag{A48}$$

where

$$d_{t,i,j,\varepsilon} = \frac{\left(e^{(\epsilon)\mathrm{x}_i\hat{s}_{t,i}}\left((\epsilon)\left(\mathrm{x}_i\hat{s}_{t,i} - \mathrm{x}_j\hat{s}_{t,j}\right) - 1\right) + e^{(\epsilon)\mathrm{x}_j\hat{s}_{t,j}}\right)}{(\epsilon)^2\left(\mathrm{x}_i\hat{s}t,i - \mathrm{x}_j\hat{s}_{t,j}\right)}$$

and where $\hat{s}_{t,i} \equiv \hat{s}_t(x_i)$ and where $\mathbb{1}_{i\neq 1}$ and $\mathbb{1}_{i\neq I}$ are indicator functions equal to 1 when i is different from 1 or I, that is whenever $\mathrm{g}_t^c(\mathrm{x}_i)$ is evaluated at the bounds of the (S, s) band. It plays a similar role as the values 0.5 at the two extremes of the vector $\mathbf{e}_t^T$ above.

Hence, we can re-write equation (A48) in matrix form as

$$e^{p_t^*(\epsilon-1)} = \mathbf{d}_{t,1-\varepsilon}^T \mathbf{g}_t^{\boldsymbol{c}} + \mathrm{g}_t^0 \tag{A49}$$

where $\mathbf{g}_t^{\boldsymbol{c}}$ is the vector collecting the values of the the distribution function g_t^c at the grid points and where the vector $\mathbf{d}_{t,1-\varepsilon}$ is

$$\mathbf{d}_{t,1-\varepsilon} = \left[\mathbb{1}_{i\neq 1}d_{t,i,i-1,1-\varepsilon} + \mathbb{1}_{i\neq I}d_{t,i,i+1,1-\varepsilon}\right]_{i=1}^{I}.$$

Here we have adopted the notation that $[x_i]_{i=1}^{I}$ denotes a $I \times 1$ vector with elements x_i.

F.3.3. Labor Market

Following the previous subsection, the labor market condition (A46) is computed as

$$N_t = \frac{C_t}{A_t} e^{p_t^*(-\epsilon)} \left(\sum_{i=1}^{I} (g_t^c(x_i) \mathbb{1}_{i \neq 1} d_{t,i,i-1,-\varepsilon} + g_t^c(x_i) \mathbb{1}_{i \neq I} d_{t,i,i+1,-\varepsilon}) + g_{t-1}^0 \right) + \eta g_{t-1}^0$$

which we re-write in matrix form as

$$N_t = \frac{C_t}{A_t} e^{p_t^*(-\epsilon)} \left(\mathbf{d}_{t,-\varepsilon}^T \mathbf{g_t^c} + g_{t-1}^0 \right) + \eta g_{t-1}^0. \tag{A50}$$

F.3.4. Distribution

Once we have evaluated the integrals, the distribution function in (A43) can be written as:

$$g_t^c(x_j) = \sum_{i=1}^{I} \frac{1}{2\sqrt{2\pi}} g_{t-1}^c(x_i) \left[\mathbb{1}_{i \neq 1} f_{t,i,i-1,j} + \mathbb{1}_{i \neq I} f_{t,i,i+1,j} \right] + \frac{1}{\sigma} g_{t-1}^0 \phi \left(\frac{-\hat{s}_{t,j} x_j - \pi_t^*}{\sigma} \right) \tag{A51}$$

where from now on, π without time subindex, denotes the scalar π, $f_{t,i,\bar{i},j}$ and $\mathcal{P}_{t,i,j}$ are defined as

$$f_{t,i,\bar{i},j} = \frac{\sqrt{2\pi} \left(\mathcal{P}_{t,\bar{i},j} \right) \left(\operatorname{erf}\left(\frac{\mathcal{P}_{t,\bar{i},j}}{\sqrt{2}\sigma} \right) - \operatorname{erf}\left(\frac{\mathcal{P}_{t,i,j}}{\sqrt{2}\sigma} \right) \right) + 2\sigma \left(\exp\left(-\frac{\mathcal{P}_{t,\bar{i},j}^{\;2}}{2\sigma^2} \right) - \exp\left(-\frac{\mathcal{P}_{t,i,j}^{\;2}}{2\sigma^2} \right) \right)}{\left| x_i \hat{s}_{t-1,i} - x_{\bar{i}} \hat{s}_{t-1,\bar{i}} \right|},$$

$$\mathcal{P}_{t,i,j} = -x_i \hat{s}_{t-1,i} + x_j \hat{s}_{t,j} + \pi_t^*.$$

For compactness, define

$$\mathbf{g_t^c} \equiv \left[g_t^c(x_j) \right]_{j=1}^{I}$$

$$\mathbf{F}_t \equiv \left[\frac{1}{2\sqrt{2\pi}} \left(\mathbb{1}_{i \neq 1} f_{t,i,i-1,j} + \mathbb{1}_{i \neq I} f_{t,i,i+1,j} \right) \right]_{j=1,i=1}^{I,I}$$

$$\mathbf{f}_t \equiv \left[\frac{1}{\sigma} \phi \left(\frac{-\hat{s}_{t,j} x_j - \pi_t^*}{\sigma} \right) \right]_{j=1}^{I}$$

where $\mathbf{g_t^c}$ and $\mathbf{f}_t$ are vectors with the probability mass function g_t^c and the scaled and shifted normal distribution at the grid points, respectively, $\mathbf{F}_t$ is a matrix that captures the idiosyncratic transitions due to firm-level quality shocks and where we have adopted the notation that $\left[x_{i,j} \right]_{j=1,i=1}^{J,I}$ denotes a $J \times I$ matrix with elements $x_{j,i}$. Thus, equation A51 can be represented in matrix form as

$$\mathbf{g}_t^c = \mathbf{F}_t \mathbf{g}_{t-1}^c + \mathbf{f}_t g_{t-1}^0. \tag{A52}$$

F.3.5. Value function

Once we have evaluated the integrals, and denoting the standard normal cdf by $\Phi(\cdot)$ and the central grid point by i_0 (i.e. for $x_{i_0} = 0$), the value function A41 can be written as

$$\begin{aligned} v_t(x_j) =\ & \Pi_{j,t} - \Pi_{j,t}(0) \\ & + \Lambda_{t,t+1} \sum_{i=1}^{I} \frac{1}{2\sqrt{2\pi}} v_{t+1}(x_i) \left(\mathbb{1}_{i \neq 1}(a_{t,i,i-1,j} - a_{t,i_0,i_0-1,j}) + \mathbb{1}_{i \neq I}(a_{t,i,i+1,j} - a_{t,i_0,i_0+1,j}) \right) \\ & + \Lambda_{t,t+1} \left(-\eta w_{t+1}\right) \left(\Phi\left(\frac{\mathcal{P}_{t+1,j,I}}{\sigma}\right) - \Phi\left(\frac{\mathcal{P}_{t+1,j,1}}{\sigma}\right) - \Phi\left(\frac{\mathcal{P}_{t+1,i_0,I}}{\sigma}\right) + \Phi\left(\frac{\mathcal{P}_{t+1,i_0,1}}{\sigma}\right) \right) \end{aligned} \tag{A53}$$

where

$$a_{t,i,\bar{i},j} = \frac{\sqrt{2\pi}\left(\mathcal{P}_{t+1,j,\bar{i}}\right)\left(\operatorname{erf}\left(\frac{\mathcal{P}_{t+1,j,\bar{i}}}{\sqrt{2}\sigma}\right) - \operatorname{erf}\left(\frac{\mathcal{P}_{t+1,j,i}}{\sqrt{2}\sigma}\right)\right) + 2\sigma\left(\exp\left(-\frac{\left(\mathcal{P}_{t+1,j,\bar{i}}\right)^2}{2\sigma^2}\right) - \exp\left(-\frac{\left(\mathcal{P}_{t+1,j,i}\right)^2}{2\sigma^2}\right)\right)}{\left| x_i \hat{s}_{t+1,i} - x_{\bar{i}} \hat{s}_{t+1,\bar{i}} \right|}. \tag{A54}$$

For compactness, let us define

$$\begin{aligned} \mathbf{v}_t &\equiv \left[v_t(x_j) \right]_{j=1}^{I}, \\ \Pi_t &\equiv \left[\Pi_{j,t} - \Pi_{j,t}(0) \right]_{j=1}^{I}, \\ \mathbf{A}_t &\equiv \left[\Lambda_{t,t+1} \frac{1}{2\sqrt{2\pi}} (\mathbb{1}_{i \neq 1}(a_{t,i,i-1,j} - a_{t,i_0,i_0-1,j}) + \mathbb{1}_{i \neq I}(a_{t,i,i+1,j} - a_{t,i_0,i_0+1,j})) \right]_{j=1,i=1}^{I,I}, \\ \mathbf{b}_{t+1} &\equiv \left[\Lambda_{t,t+1} \left(\Phi\left(\frac{\mathcal{P}_{t+1,j,I}}{\sigma}\right) - \Phi\left(\frac{\mathcal{P}_{t+1,j,1}}{\sigma}\right) - \Phi\left(\frac{\mathcal{P}_{t+1,i_0,I}}{\sigma}\right) + \Phi\left(\frac{\mathcal{P}_{t+1,i_0,1}}{\sigma}\right) \right) \right]_{j=1}^{I} \end{aligned}$$

where $\mathbf{v}_t$ and $\mathbf{b}_{t+1}$ are vectors that evaluate the value function v_t and the adjustment probability at different grid points, Π_t is the vector of profit differences, while $\mathbf{A}_t$ is a matrix that represents the idiosyncratic transition due to firm-level quality shocks and price updating. Thus, equation (A53) can be represented in matrix form as

$$\mathbf{v}_t = \Pi_t + [\mathbf{A}_t \mathbf{v}_{t+1} - \mathbf{b}_{t+1} \eta w_{t+1}]. \tag{A55}$$

F.3.6. Optimality condition for the reset price

After evaluating the integral, we can write the optimality condition in (A42) as

$$\begin{aligned} 0 &= \Pi_t'(0) + \Lambda_{t,t+1} \sum_{i=1}^{I} \mathrm{v}_{t+1}(\mathrm{x_i}) \frac{1}{2} \left(\mathbb{1}_{i \neq 1} c_{t,i,i-1,i_0} + \mathbb{1}_{i \neq I} c_{t,i,i+1,i_0} \right) \\ &+ \frac{\Lambda_{t,t+1}}{\sigma} \left(\phi\left(\frac{-S_{t+1} - \pi^*_{t+1}}{\sigma} \right) - \phi\left(\frac{-s_{t+1} - \pi^*_{t+1}}{\sigma} \right) \right) (-\eta w_{t+1}) \end{aligned} \tag{A56}$$

where

$$c_{t,i,\bar{i},j} = \frac{\operatorname{erf}\left(\frac{\mathcal{P}_{t+1,j,i}}{\sqrt{2}\sigma}\right) - \operatorname{erf}\left(\frac{\mathcal{P}_{t+1,j,\bar{i}}}{\sqrt{2}\sigma}\right)}{x_i \hat{s}_{t+1,i} - x_{\bar{i}} \hat{s}_{t+1,\bar{i}}} - \frac{\sqrt{\frac{2}{\pi}} \exp\left(- \frac{\left(\mathcal{P}_{t+1,j,i}\right)^2}{2\sigma^2} \right)}{\sigma}. \tag{A57}$$

We can write this equation using matrix notation:

$$\begin{aligned} 0 &= \Pi_t'(0) + \mathbf{c}_{t+1}^T \mathbf{v}_{t+1} \\ &+ \frac{\Lambda_{t,t+1}}{\sigma} \left(\phi\left(\frac{-S_{t+1} - \pi^*_{t+1}}{\sigma} \right) - \phi\left(\frac{-s_{t+1} - \pi^*_{t+1}}{\sigma} \right) \right) (-\eta w_{t+1}) \end{aligned} \tag{A58}$$

where

$$\mathbf{c}_{t+1} = \left[\Lambda_{t,t+1} \frac{1}{2} \left(\mathbb{1}_{i \neq 1} c_{t,i,i-1,i_0} + \mathbb{1}_{i \neq I} c_{t,i,i+1,i_0} \right) \right]_{i=1}^{I}. \tag{A59}$$

F.4. Final equation system

Collecting the derived equations, and combining them with the remainder of the private equilibrium conditions (which contain no infinite dimensional objects) and the objective, we can approximate the infinite dimensional central bank problem by the following finite dimensional problem:

$$\max_{\left\{ \mathbf{g}_t^{\mathbf{c}}, \mathrm{g}_t^0, \mathbf{v}_t, C_t, w_t, p_t^*, s_t, S_t, \pi_t^* \right\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \left(\log C_t - \left(\frac{C_t}{A_t} e^{p_t^*(-\epsilon)} \left(\mathbf{d}_{t,-\varepsilon}^T \mathbf{g}_t^{\mathbf{c}} + \mathrm{g}_{t-1}^0 \right) + \eta \mathrm{g}_{t-1}^0 \right) \right)$$

subject to

$$\begin{aligned} w_t &= C_t, \\ \mathbf{v}_t &= \Pi_t + \mathbf{A}_t \mathbf{v}_{t+1} - \mathbf{b}_{t+1} \eta w_{t+1}, \\ \mathbf{v}_{t,1} &= -\eta w_t, \\ \mathbf{v}_{t,I} &= -\eta w_t, \\ 0 &= \Pi_t'(0) + \mathbf{c}_{t+1}^T \mathbf{v}_{t+1} + \frac{\Lambda_{t,t+1}}{\sigma} \left(\phi\left(\frac{-S_{t+1} - \pi^*_{t+1}}{\sigma} \right) - \phi\left(\frac{-s_{t+1} - \pi^*_{t+1}}{\sigma} \right) \right) (-\eta w_{t+1}), \\ \mathbf{g}_t^{\mathbf{c}} &= \mathbf{F}_t \mathbf{g}_{t-1}^{\mathbf{c}} + \mathbf{f}_t \mathrm{g}_{t-1}^0, \\ \mathrm{g}_t^0 &= 1 - \mathbf{e}_t^T \mathbf{g}_t^{\mathbf{c}}, \\ e^{p_t^*(\epsilon - 1)} &= \mathbf{d}_{t,1-\varepsilon}^T \mathbf{g}_t^{\mathbf{c}} + \mathrm{g}_t^0. \end{aligned}$$

Here, the choice variables $\mathbf{v}_t$ and $\mathbf{g}_t^{\mathbf{c}}$ are vectors of length I. The rest of the choice variables are scalars. Note that the choice variables p_t^*, s_t, S_t, π_t^* implicitly appear in the problem (inside the vectors and matrices $\mathbf{A}_t$, $\mathbf{b}_t$, etc.)

As already explained at the beginning of this Appendix, we solve for the FOCs of this system by symbolic differentiation. The resulting system of FOCs is then solved in the sequence space. We next explain how we find the steady state, which serves as initial and terminal condition for dynamic simulations.

F.5. Steady state

To solve for the steady state of the private equilibrium conditions, given a policy $\bar{\pi}$, the algorithm is as follows. We rely on steady-state relationships $w = C$, and $R = (1+\pi)/\beta$ and $\pi = \pi^*$. We start with a guess for the real wage w, the optimal rest price p^*, and the bounds of the (S, s) band s and S then:

a. Compute consumption $C = w$.
b. Using $\pi = \pi^* = \bar{\pi}$, C and the 4 initial guesses, solve for that stationary value function using the Bellman equation and the stationary distribution using the law of motion of the distribution. Both have closed-form solutions given the guesses.

$$\begin{aligned} \mathbf{v} &= (\mathbf{I} - \mathbf{A})^{-1} (\Pi - \mathbf{b}\eta w), \\ \mathbf{g}^c &= \left(\mathbf{I} - \mathbf{F} + \mathbf{f}\mathbf{e}^T\right)^{-1} \mathbf{f}, \\ g^0 &= 1 - \mathbf{e}^T \mathbf{g}^c \end{aligned}$$

c. Compute the residuals of the 4 remaining equations

$$\begin{aligned} \mathbf{v}_{t,1} &= -\eta w_t, \\ \mathbf{v}_{t,I} &= -\eta w_t, \\ 0 &= \Pi_t'(0) + \mathbf{c}_{t+1}^T \mathbf{v}_{t+1} + \frac{\Lambda_{t,t+1}}{\sigma} \left(\phi\left(\frac{-S_{t+1} - \pi_{t+1}^*}{\sigma}\right) - \phi\left(\frac{-s_{t+1} - \pi_{t+1}^*}{\sigma}\right) \right) (-\eta w_{t+1}), \\ e^{p_t^*(\epsilon - 1)} &= \mathbf{h}_{t,1-\varepsilon}^T \mathbf{g}_t^{\mathbf{c}} + g_t^0. \end{aligned}$$

d. Use a Newton method to update the 4 guesses (w, p^*, s, S) and return to step 1, until convergence of the residuals.

Appendix G. The CalvoPlus model

The setup follows very closely Section 2, so we introduce minimal modifications to the notation. The menu cost now is a random variable $\tilde{\eta}$ such that

$$\tilde{\eta} = \begin{cases} \eta & \text{with prob } \alpha \\ 0 & \text{with prob } 1 - \alpha \end{cases}$$

so the probability that a price p is adjusted is

$$\Omega_t(p) = Pr[\tilde{\eta} = 0] + Pr[\tilde{\eta} = \eta]\lambda_t(p) = (1-\alpha) + \alpha\lambda_t(p).$$

The function $\lambda_t(p)$ is the probability of a price change conditional on the menu cost being η:

$$\lambda_t(p) = 1[L(p) > 0]$$

where the difference in value between adjusting and not adjusting the price must be higher than the menu cost – which is expressed in terms of labor cost:

$$L(p) = \max_{p'} V_t(p') - \eta w_t - V(p).$$

With that, the firm's value function now is

$$\begin{aligned} V_t(p) &= \Pi(p, w_t, A_t) \\ &+\alpha\mathbb{E}_t\left[(1-\lambda_{t+1}(p - \sigma_{t+1}\varepsilon_{t+1} - \pi_{t+1}))\Lambda_{t,t+1}V_{t+1}(p - \sigma\varepsilon_{t+1} - \pi_{t+1})\right] \\ &+\alpha\mathbb{E}_t\left[\lambda_{t+1}(p - \sigma_{t+1}\varepsilon_{t+1} - \pi_{t+1})\Lambda_{t,t+1}\left(\max_{p'} V_{t+1}(p') - \eta w_{t+1}\right)\right] \\ &+(1-\alpha)\mathbb{E}_t\left[\Lambda_{t,t+1}\left(\max_{p'} V_{t+1}(p')\right)\right] \end{aligned}$$

which accounts for the fact that with probability $1-\alpha$ the price can be adjusted for free. As the menu cost is expressed in labor units, the labor market clearing condition in equation (23) in Section 2 must be modified to

$$N_t = \frac{C_t}{A_t}\int e^{p(-\epsilon)} g_t(p)\,dp + \alpha\eta\int \lambda_t(p - \sigma\varepsilon_t - \pi_t)\,g_{t-1}(p)\,dp$$

such that a share α of firms for which it is worthwhile to incur the menu cost η actually pay it. Note that now the frequency of price changes is given by

$$f_t = \int \Phi_t(p)\,g_{t-1}(p)\,dp = (1-\alpha) + \alpha\int \lambda_t(p - \sigma\varepsilon_t - \pi_t)\,g_{t-1}(p)\,dp.$$

The next equation to modify is the law of motion of the price density function:

$$\begin{aligned} g_t(p) &= \alpha(1-\lambda_t(p))\int g_{t-1}(p + \sigma\varepsilon_t + \pi_t)\,d\xi(\varepsilon) \\ &+\delta(p - p_t^*)\int [(1-\alpha) + \alpha\lambda_t(\tilde{p})]\left(\int g_{t-1}(\tilde{p} + \sigma\varepsilon_t + \pi_t)\,d\xi(\varepsilon)\right)d\tilde{p}. \end{aligned}$$

Summing up, the objective of the Ramsey problem in Section 3.1 now is

$$\max_{\substack{\{g_t^c(\cdot), g_t^0, V_t(\cdot), C_t,\\ w_t, p_t^*, s_t, S_t, \pi_t^*\}_{t=0}^{\infty}}} \sum_{t=0}^{\infty}\beta^t\left(\log C_t - \frac{C_t}{A_t}\left(\int e^{(x+p_t^*)(-\epsilon_t)} g_t^c(p)\,dx + g_t^0 e^{(p_t^*)(-\epsilon)}\right) - \eta[g_t^0 - (1-\alpha)]\right)$$

subject to

$$
\begin{aligned}
w_t &= C_t, \\
V_t(x) &= \Pi\left(x, p_t^*, w_t, A_t\right) + \alpha \frac{\Lambda_{t,t+1}}{\sigma} \int_{s_t}^{S_t} \left[V_{t+1}\left(x'\right) \phi\left(\frac{\left(x - x'\right) - \pi_{t+1}^*}{\sigma} \right) \right] dx' + \\
&\quad + \alpha \Lambda_{t,t+1} \left(1 - \frac{1}{\sigma} \int_{s_t}^{S_t} \phi\left(\frac{\left(x - x'\right) - \pi_{t+1}^*}{\sigma} \right) dx' \right) \left[V_{t+1}(0) - \eta w_{t+1} \right] \\
&\quad + (1 - \alpha) V(0), \\
V_t(s_t) &= V_t(0) - \eta w_t, \\
V_t(S_t) &= V_t(0) - \eta w_t, \\
V_t'(0) &= 0, \\
g_t^c(x) &= \frac{\alpha}{\sigma} \int_{s_{t-1}}^{S_{t-1}} g_{t-1}^c(x_{-1}) \phi\left(\frac{(x_{-1} - x) - \pi_t^*}{\sigma} \right) dx_{-1} + \alpha g_{t-1}^0 \phi\left(\frac{-x - \pi_t^*}{\sigma_t} \right), \\
g_t^0 &= 1 - \int_{s_t}^{S_t} g_t^c(x)\, dx, \\
1 &= \int e^{\left(x + p_t^*\right)(1 - \epsilon)} g_t^c(x)\, dx + g_t^0 e^{p_t^*(1 - \epsilon)}.
\end{aligned}
$$

Acknowledgements
We are grateful to Guido Ascari, Vladimir Asryan, Andres Blanco, Davide Debortoli, Eduardo Engel, Aurélien Eyquem, Jordi Galí, Erwan Gautier, Mark Gertler, Mishel Ghassibe, Basil Halperin, Francesco Lippi, Albert Marcet, Alberto Martin, Virgiliu Midrigan, Giorgio Primiceri, Xavier Ragot, Morten Ravn, Tom Sargent, Edouard Schaal, Raphael Schoenle, Mathias Trabandt and Jaume Ventura, as well as to participants at various conferences and seminars for their comments and suggestions.
The views expressed here are those of the authors only and do not necessarily represent those of the Bank of Spain, the Central Bank of Chile, the ECB, or the Eurosystem. Previous version has been circulated under the subtitle “Optimal Monetary Policy under a Nonlinear Phillips Curve”.

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PDF ISBN 978-92-899-7379-3 ISSN 1725-2806 doi: 10.2866/1683689 QB-01-25-155-EN-N